Arithmetic of Hecke eigenvalues of Siegel modular forms

By

Biplab Paul

MATH10201404006

The Institute of Mathematical Sciences, Chennai

A thesis submitted to the

Board of Studies in Mathematical Sciences

In partial fulfillment of requirements

for the Degree of

DOCTOR OF PHILOSOPHY

of

HOMI BHABHA NATIONAL INSTITUTE

April, 2019

Homi Bhabha National Institute

Recommendations of the Viva Voce Committee

As members of the Viva Voce Committee, we certify that we have read the dissertation prepared by Biplab Paul titled "Arithmetic of Hecke eigenvalues of Siegel modular forms" and recommend that it may be accepted as fulfilling the thesis requirement for the award of Degree of Doctor of Philosophy.

Nagaraj. D. S.
Chairman - D. S. Nagaraj — Date: April 26, 2019

Sanoli Gun
Guide/Convenor - Sanoli Gun — Date: April 26, 2019

Stephan Baier
Examiner - Stephan Baier — Date: April 26, 2019

K. Vijay.
Member 1 - Vijay Kodiyalam — Date: April 26, 2019

Anirban
Member 2 - Anirban Mukhopadhyay — Date: April 26, 2019

Purusottam Rath
Member 3 - Purusottam Rath — Date: April 26, 2019

Final approval and acceptance of this thesis is contingent upon the candidate's submission of the final copies of the thesis to HBNI.

I hereby certify that I have read this thesis prepared under my direction and recommend that it may be accepted as fulfilling the thesis requirement.

Date: April 26, 2019

Place: Chennai

Sanoli Gun
Guide

STATEMENT BY AUTHOR

This dissertation has been submitted in partial fulfillment of requirements for an advanced degree at Homi Bhabha National Institute (HBNI) and is deposited in the Library to be made available to borrowers under rules of the HBNI.

Brief quotations from this dissertation are allowable without special permission, provided that accurate acknowledgement of source is made. Requests for permission for extended quotation from or reproduction of this manuscript in whole or in part may be granted by the Competent Authority of HBNI when in his or her judgement the proposed use of the material is in the interests of scholarship. In all other instances, however, permission must be obtained from the author.

Biplab Paul
26.04.2019

Biplab Paul

DECLARATION

I hereby declare that the investigation presented in the thesis has been carried out by me. The work is original and has not been submitted earlier as a whole or in part for a degree / diploma at this or any other Institution / University.

Biplab Paul
26.04.2019
Biplab Paul

LIST OF PUBLICATIONS ARISING FROM THE THESIS

Journal

1. "On Hecke eigenvalues of Siegel modular forms in the Maass space", Sanoli Gun, Biplab Paul and Jyoti Sengupta, Forum Math., **30** (2018), 775–783.

2. "The first simultaneous sign change and non-vanishing of Hecke eigenvalues of newforms", Sanoli Gun, Balesh Kumar and Biplab Paul, To appear in J. Number Theory

Submitted

1. "Simultaneous arithmetic behaviour of Hecke eigenvalues of Siegel cusp forms of degree two", Sanoli Gun, Winfried Kohnen and Biplab Paul.

2. "Sign changes of Fourier coefficients of newforms and multiplicity one theorem", Sanoli Gun and Biplab Paul.

Biplab Paul
26.04.2019

Biplab Paul

Dedicated to my Teachers ...

"Guru Brahma, Guru Vishnu, Guru Devo Maheshwarah;

Guru Sakshat Param Brahma Tasmai Sri Guruve Namaha."

ACKNOWLEDGEMENTS

It is my great pleasure to have this opportunity to express my thanks and gratitude to my advisor Prof. Sanoli Gun for her advice, encouragement and constant support during my Ph.D. days. This thesis would not have been possible without her advice and careful guidance. I am fortunate to have Sanoli di as my friend, philosopher and guide. I would also like to express my gratitude to Prof. Purusottam Rath from whom I learnt a lot. He has always been a kind and illuminating teacher. I would like to take this opportunity to thank all my coauthors Prof. Sanoli Gun, Dr. Balesh Kumar, Prof. Winfried Kohnen and Prof. Jyoti Sengupta for some fruitful collaborations. I am also grateful to Prof. Denis Benois, Prof. M. Ram Murty, Prof. V. Kumar Murty, Prof. Joseph Oesterlé and Prof. Patrice Philippon for their exciting courses.

I would like to thank all the faculty members of IMSc for the courses they have given. I thank the members of my doctoral committee, Prof. Sanoli Gun, Prof. Vijay Kodiyalam, Prof. Anirban Mukhopadhyay, Prof. D. S. Nagaraj and Prof. Purusottam Rath for their constant support. I am grateful to IMSc and DAE for providing financial assistance through out my Phd and for providing excellent academic atmosphere and facilities. I thank library staff and administrative members of IMSc for making our life easier and comfortable. I also would like to extend thanks to all the canteen staff, housekeeping people and gardeners for making my stay at IMSc very enjoyable.

I feel blessed to have many teachers who were there with me in the journey towards my Ph. D. To start with Biswajit Daradi Sir, Dipak Ghosh Sir, Gurugobinda Sarkar Sir, Manash Sir, Nirmal Roy Sir, Nityananda Sir, Sanjay Sir, Suranjan Sir, Swapan Sir and Hridoy Sir from my school days, then AG Sir, DG Sir, GR Sir, MNM Sir and RM Sir from my B. Sc. life and finally AA Sir, MNM Sir, SJ Sir, SKA Sir from my masters studies. Without their support, love and advice, it would have been very difficult for me to continue my study. I take this opportunity to convey my heartfelt gratitude and respect to all of them.

This note of acknowledgement would not be complete without mentioning the support from Dr. B. K. Bhowmick, Chanchal Majumder, Lily Foundation, Priyamahonto Debnath and others at various crucial stages of my life. I am indebted to all of them for their support and unconditional love. I also would like to thank WB state Govt.'s merit-cum-means scholarship scheme and NBHM MSc fellowship scheme for supporting my studies in respective tenures; without these fellowships I would not be here today.

Starting from school life till the end of Ph. D., I am lucky to have some good friends. I have spent eight years or more with some of my friends like Biswajit, Sribas, Palash, Krishna, Mousumi, Nabamita, Sathi, Sanjoy, "Choto vai". I would not forget the company of Aditi, Buddha, Jyoti, Manideepa, Roghu, Sagar, Siddhartha, Subhankar, Sumit. I was fortunate to have some good company at IMSc from Pranendu, Rupam, Snehajit da, Nabanita, Sruthi, Dijoy, Anupam, Abinash, Kasi. I would like to thank all of them. A special thanks to Abhishek and Jyothsnaa for many academic discussions and lot of supports. Also a special note of thanks to my seniors Biswa da, Ekata di and Tapas da for their support.

Words are not enough to convey my gratitude and love to my parents and my brothers whose unconditional love and support helped me to complete this thesis. I record my thank to my boromama, mejomama, chotomama, dadu and dida for supporting me in various ways. During my Ph.D. days, both of my brothers have successfully taken care of all those responsibilities what I was supposed to do. Without their support it would not have been possible for me to stay away from my home and complete my thesis. It is my pleasure to thank them.

Finally, I would like to bow down to Almighty for all these.

Contents

Synopsis

The thesis centers around the following question:

Question. Given two Siegel cusp forms F and G of same degree and weights k_1 and k_2 respectively, how do we determine whether $F = G$?

For elliptic modular forms which are Siegel modular forms of degree one, the above question has been addressed by several authors, namely W. Kohnen, J. Sengupta [39], E. Kowalski, Y. K. Lau, K. Soundararajan, J. Wu [41], W. Luo [52], D. Ramakrishnan [51], K. Matomäki [57], M. R. Murty [64] among others. In this doctoral thesis, we address the above question by appealing to the arithmetic properties of the Hecke eigenvalues of Siegel cusp forms of degree one and two. In particular, our focus is to derive bounds for these eigenvalues, their non-vanishing as well as quantitative sign changes and to exploit these as essential tools to distinguish cusp forms of degree one and two.

Let us begin by setting up the notions and notations relevant to our purpose. For integers $g \geq 1$ and $k \geq 0$, let $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z})$ be the Siegel modular group of degree g and $S_k(\Gamma_g)$ be the space of cuspidal Siegel modular forms of weight k and degree g for Γ_g. For a positive integer n, recall the n-th Hecke operator $T_g(n)$ on the space $S_k(\Gamma_g)$ is given by

$$T_g(n)F := n^{gk-\frac{g(g+1)}{2}} \sum_{\gamma \in \Gamma_g \backslash \mathcal{O}_{g,n}} F \mid \gamma,$$

where

$$\mathcal{O}_{g,n} := \left\{\gamma \in M_{2g}(\mathbb{Z}) \mid \gamma^t J\gamma = nJ\right\}, \quad J := \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

It is known that the complex vector space $S_k(\Gamma_g)$ has a basis consisting of eigenvectors of all the Hecke operators $T_g(n)$. Let $F \in S_k(\Gamma_g)$ be such an eigenvector of $T_g(n)$ with eigenvalue $\mu_F(n)$, that is, $T_g(n)F = \mu_F(n)F$ for all $n \in \mathbb{N}$. Then one knows that μ_F is a multiplicative function.

Note that for degree $g = 1$, the space $S_k(\Gamma_1)$ is nothing but the space of elliptic cusp forms of level 1 and weight k. In this case, by a celebrated work of P. Deligne, one knows that the Ramanujan-Petersson conjecture is true, i.e. for any prime p, one has

$$|\mu_F(p)| \le 2p^{(k-1)/2}.$$

One natural question to ask is whether this upper-bound is optimal. In other words, one asks for an Omega result. In 1983, M. R. Murty [64] proved that

$$\mu_F(n) = \Omega_\pm\left(n^{(k-1)/2}\exp\left(\frac{c\log n}{\log\log n}\right)\right)$$

for some positive constant c. Here for any arithmetic functions f and g with $g(n) > 0$ for all $n \in \mathbb{N}$, the symbol $f(n) = \Omega_\pm(g(n))$ means

$$\limsup_{n\to\infty}\frac{f(n)}{g(n)} > 0 \quad \text{and} \quad \liminf_{n\to\infty}\frac{f(n)}{g(n)} < 0.$$

Note that this also shows that the sequence $\{\mu_F(n)\}_{n\in\mathbb{N}}$ changes sign infinitely often. The recent works of E. Kowalski, Y. K. Lau, K. Soundararajan and J. Wu [41] and of K. Matomäki [57] prove that any normalized Hecke eigenform $f \in S_k^{new}(N)$ is uniquely determined by the signs of its Hecke eigenvalues at primes. Here $S_k^{new}(N)$ denotes the space of newforms of weight k for $\Gamma_0(N)$.

We now briefly describe our results in this context. In [28], we investigate simultaneous sign change and non-vanishing of Hecke eigenvalues of newforms which are normalized Hecke eigenforms. More precisely, for $z \in \mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $q := e^{2\pi i z}$, let

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_{k_1}^{new}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n) q^n \in S_{k_2}^{new}(N_2) \tag{0.0.1}$$

be normalized Hecke eigenforms. In this case, normalization ensures that

$$\mu_f(n) = a_f(n) \quad \text{and} \quad \mu_g(n) = a_g(n)$$

for all $n \in \mathbb{N}$. Then we have the following theorem.

Theorem 0.0.1. *Let N_1, N_2 be square-free, $N := lcm[N_1, N_2]$ and $f \in S_{k_1}^{new}(N_1)$, $g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with Fourier expansions as in (0.0.1). Then there exists a prime power p^α, $\alpha \le 2$ with*

$$p^\alpha \ll_\epsilon \max\left\{\exp\left(c \log^2(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)})\right), \left[N^2\left(1 + \frac{|k_2 - k_1|}{2}\right)\left(\frac{k_1 + k_2}{2}\right)\right]^{1+\epsilon}\right\}$$

such that $a_f(p^\alpha) a_g(p^\alpha) < 0$. Here $c > 0$ is an absolute constant and $\mathfrak{q}(f)$, $\mathfrak{q}(g)$ are analytic conductors of the Rankin-Selberg L-functions of f and g respectively. Note that

$$\mathfrak{q}(f) \ll k_1^2 N_1^2 \log\log N_1 \quad \text{and} \quad \mathfrak{q}(f) \ll k_2^2 N_2^2 \log\log N_2.$$

This can be thought of as sign change analogue of the classical Sturm's bound. A. Ghosh and P. Sarnak [24], in their study of distribution of real zeros of Hecke eigenforms, relate the question of sign changes of Fourier coefficients of Hecke eigenforms to the question of distribution of real zeros of those forms. In a recent work [30] with S. Gun, we relate the question of simultaneous sign changes of Fourier coefficients of primitive cusp forms to multiplicity one theorem for those forms. More precisely, for any $f \in S_k(N)$ which is a normalized Hecke eigenform with Fourier coefficients $a_f(n)$, let us set

$$\lambda_f(n) := \frac{a_f(n)}{n^{(k-1)/2}}.$$

With this notation in place, we show the following.

Theorem 0.0.2. *Let $f \in S_{k_1}(N_1)$ and $g \in S_{k_2}(N_2)$ be normalized Hecke eigenforms and p be a prime such that $(p, N_1N_2) = 1$. Then the following conditions are equivalent;*

- *there exist infinitely many $m \geq 1$ such that $\lambda_f(p^m)\lambda_g(p^m) > 0$ and infinitely many $m \geq 1$ such that $\lambda_f(p^m)\lambda_g(p^m) < 0$;*
- *one has $\lambda_f(p) \neq \lambda_g(p)$.*

This theorem allows us to estimate the density of the set of primes p for which the sequence $\{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ changes sign infinitely often.

In order to state our next theorem, we shall need the notion of CM forms which we recall.

Definition 0.0.3. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform.*

- *We say that f has complex multiplication (or of CM type) if there exists a non-trivial Dirichlet character χ modulo D such that*

  $$a_f(p)\chi(p) = a_f(p)$$

  for all primes p lying in a set of density 1.
- *A form is called a non-CM form or of non-CM type if it is not of CM type.*

We also need the following notion of natural density of a subset of the set of primes.

Definition 0.0.4. *Let A be a subset of the set of primes $\mathcal{P}$. We say that the natural density of the set A is $d(A)$ if the limit*

$$\lim_{x \to \infty} \frac{\#\{p \le x \mid p \in A\}}{\#\{p \le x \mid p \in \mathcal{P}\}}$$

exists and is equal to $d(A)$.

Let $f \in S_k^{new}(N)$ be a CM form. Then by a work of K. Ribet [82], one knows that there exists a Hecke character χ of an imaginary quadratic field K such that the Fourier coefficients of f are determined by χ. In this case, we shall say that the form f has CM by the imaginary quadratic field K. Now by applying Theorem 0.0.2, we have

Theorem 0.0.5. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms with Fourier expansions as in* (0.0.1) *and S be the set of primes p for which the sets*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) > 0\} \quad \textit{and} \quad \{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) < 0\}$$

are infinite. Then,

1. *if at least one of f or g is a non-CM form, then*
   - *the natural density of S is 1 provided $f \neq g \otimes \chi$ for any Dirichlet character χ;*
   - *the natural density of S is $1/2$ if $f = g \otimes \chi$ for some Dirichlet character χ.*
2. *if both f and g are of CM type, then*
   - *the lower natural density of S is greater than or equal to $1/2$ if either $k_1 \neq k_2$ or f and g have CM by different quadratic fields;*

- *the lower natural density of S is greater than or equal to $1/8$ if $k_1 = k_2$ and the forms f and g have CM by the same field.*

Theorem 0.0.5 improves a recent result of S. Gun, W. Kohnen and P. Rath [27, Theorem 3]. If we assume that at least one of f or g is non-CM, then we can prove the following stronger result.

Theorem 0.0.6. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms with Fourier expansions as in (0.0.1) and not both of CM type. For any positive integer j, let S_j be the set of primes p such that*

$$\left\{m \in \mathbb{N} \mid a_f(p^{jm})a_g(p^{jm}) > 0\right\} \quad \text{and} \quad \left\{m \in \mathbb{N} \mid a_f(p^{jm})a_g(p^{jm}) < 0\right\}$$

are infinite. Then,

1. *if $f \neq g \otimes \chi$ for any Dirichlet character χ, then the natural density of S_j is equal to one for any $j \in \mathbb{N}$.*
2. *when $f = g \otimes \chi$ for some Dirichlet character χ, then*
   - *if j is odd, then the natural density of S_j is equal to $1/2$;*
   - *if j is even, then the natural density of S_j is equal to zero.*

The above theorem can be thought of as a generalization of the following result of W. Kohnen and Y. Martin [40].

Theorem 0.0.7. [W. Kohnen and Y. Martin] *Let $f \in S_k(1)$ be a normalized Hecke eigenform. Then for any integer $j \geq 1$ and for almost all primes p, the sequence $\{a_f(p^{nj})\}_{n\in\mathbb{N}}$ changes sign infinitely often.*

As mentioned in Remark 3.1 of [17], their proof is not complete for natural numbers j that are divisible by 4. In [30], along with S. Gun we proved the following theorem.

Theorem 0.0.8. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform and $j \geq 1$ be a natural number. Consider the set S_j of primes p for which the sets*

$$\{m \in \mathbb{N} \mid a_f(p^{jm}) > 0\} \quad \text{and} \quad \{m \in \mathbb{N} \mid a_f(p^{jm}) < 0\}$$

are infinite. Then,

1. *if f is a non-CM form, then the natural density of S_j is 1;*
2. *if f is of CM type and*
   - *$4|j$, then the natural density of S_j is $1/2$;*
   - *$4 \nmid j$, then the natural density of S_j is 1.*

The following theorem helps us to derive the previous one.

Theorem 0.0.9. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform and j be a positive integer. Then for almost all primes p, the following conditions are equivalent;*

1. *there exists infinitely many natural numbers $m \geq 1$ such that $\lambda_f(p^{jm}) > 0$ and infinitely many natural numbers $m \geq 1$ such that $\lambda_f(p^{jm}) < 0$;*
2. *one has*

$$\lambda_f(p) \notin \begin{cases} \{2\} & \text{for } j \text{ is odd;} \\ \{2, -2\} & \text{for } j \equiv 2 \ (mod\ 4); \\ \{-2, 0, 2\} & \text{for } j \equiv 0 \ (mod\ 4). \end{cases}$$

Further, when $k \geq 4$ or $j = 1$, then the above equivalence is true for all primes p with $(p, N) = 1$.

We now address the question of non-vanishing of Hecke eigenvalues of newforms. The study of non-vanishing of Hecke eigenvalues of newforms is inspired by the folklore conjecture of D. H. Lehmer which predicts that $\tau(n) \neq 0$ for all $n \in \mathbb{N}$. Here τ is the Ramanujan τ-function defined by the formal identity:

$$\sum_{n=1}^{\infty} \tau(n) x^n := x \prod_{n=1}^{\infty} (1 - x^n)^{24}.$$

It is well known that the above formal sum determines the unique normalized Hecke eigenform of weight 12 of degree one for the full modular group. One of the most notable results in this direction is due to J-P. Serre [91, 92] which states that the set of primes p such that $\tau(p) = 0$ has natural density zero. In fact, his result (see also [91]) characterizes non-CM forms as follows: a Hecke eigen newform is non-CM if and only if the set of primes p for which the p-th Hecke eigenvalue vanishes has natural density zero.

While the above results on sign changes of Hecke eigenvalues imply non-vanishing of the same, one can prove stronger results in this direction. In a joint work with S. Gun and B. Kumar [28], we investigate the non-vanishing nature of the sequence $\{a_f(p^m)a_g(p^m)\}_{m \in \mathbb{N}}$. Our first result in this set-up is the following.

Theorem 0.0.10. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with Fourier expansion as in* (0.0.1). *Then for all primes p with $(p, N_1N_2) = 1$, the set*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0\}$$

has positive density.

We now state our next theorem which strengthens a recent result (namely Theorem 1.2) of M. Kumari and M. R. Murty [45].

Theorem 0.0.11. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized non-CM Hecke eigenforms with Fourier expansion as in* (0.0.1). *Then there exists a set S of primes with natural density one such that for any $p \in S$ and integers $m, m' \geq 1$,* *we have*

$$a_f(p^m)a_g(p^{m'}) \neq 0.$$

Now we address the question of first simultaneous non-vanishing, analogous to that considered in Theorem 0.0.1. In particular, we have the following theorem.

Theorem 0.0.12. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with Fourier expansion as in* (0.0.1)*. Also assume that $N :=$ lcm $[N_1, N_2] > 12$. Then there exists a positive integer $1 < n \leq (2\log N)^4$ with $(n, N) = 1$ such that*

$$a_f(n)a_g(n) \neq 0.$$

Further, when N is odd, then there exists an integer $1 < n \leq 16$ with $(n, N) = 1$ such that

$$a_f(n)a_g(n) \neq 0.$$

Note that while $a_f(1)a_g(1) = 1$, our goal is to find the first $n > 1$ with $(n, N) = 1$ for which $a_f(n)a_g(n) \neq 0$; in other words to determine the first non-trivial simultaneous non-vanishing.

Now we investigate similar questions for higher degree Siegel cusp forms. In this direction the generalized Ramanujan-Petersson conjecture, as formulated in [73] implies that for any prime p and $\epsilon > 0$, one has

$$|\mu_F(p)| \ll_{g,\epsilon} p^{gk/2-g(g+1)/4+\epsilon}.$$

However when $g = 2$, this is known to hold for all Hecke eigenforms except for those lying in the Maass subspace S_k^* of $S_k(\Gamma_2)$. In fact, the Hecke eigenforms in S_k^* do not satisfy the above predicted Ramanujan-Petersson bound. This is a deep result due to R. Weissauer [99].

The above result motivates us to study the arithmetic properties such as bounds, growth, non-vanishing nature and distribution of Hecke eigenvalues of the eigenforms which are in the Maass subspace S_k^* of $S_k(\Gamma_2)$ and hence inaccessible vis-a-vis Ramanujan-Petersson bounds.

In a joint work with S. Gun and J. Sengupta [31], we proved the following theorems.

Theorem 0.0.13. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there exists an absolute constant $c > 0$ such that*

$$\mu_F(n) = \Omega\left(n^{k-1} exp\left(c\frac{\sqrt{\log n}}{\log\log n}\right)\right).$$

This gives an improvement of an earlier result of S. Das and J. Sengupta [16]. We next show that the above Omega result is not too far from an upper bound one can derive. In particular, we have the following theorem.

Theorem 0.0.14. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there exists an absolute constant $c_1 > 0$ such that*

$$\mu_F(n) \ \leq \ n^{k-1} exp\left(c_1\sqrt{\frac{\log n}{\log\log n}}\right)$$

for all $n \in \mathbb{N}$ with $n \geq 3$.

Theorem 0.0.14 improves an earlier result of A. Pitale and R. Schmidt (see page 101 of [75]). We also prove the following lower bound.

Theorem 0.0.15. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there exist absolute constants $c_2, c_3 > 0$ such that*

$$\mu_F(n) \geq c_2 n^{k-1} exp\left(-c_3\sqrt{\frac{\log n}{\log\log n}}\right)$$

for all positive integers $n \geq 3$.

As a corollary, we derive the following result of S. Breulmann [12] whose proof is rather different from ours.

Corollary 0.0.16. *If $F \in S_k^*$ is a non-zero Hecke eigenform with Hecke eigenvalues $\mu_F(n)$, then $\mu_F(n) > 0$.*

Note that $\mu_F(n)/n^{k-1} > 0$. One might wonder whether it is possible to improve the above lower bound, that is, whether there exists a real number $c > 0$ such that $\mu_F(n)/n^{k-1} > c$ for all $n \in \mathbb{N}$. Our next theorem precludes such a possibility.

Theorem 0.0.17. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then we have*

$$\liminf_{n\to\infty} \frac{\mu_F(n)}{n^{k-1}} = 0.$$

In particular, 0 is a limit point of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$. This motivated us to investigate the set of limit points of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$. In this direction, we have the following result.

Theorem 0.0.18. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there are infinitely many limit points of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$ in $(1, \infty)$ and infinitely many limit points in $(0, 1)$.*

This summarizes our study of arithmetic properties of Hecke eigenvalues of eigenforms lying in the Maass space S_k^*. Unlike the Hecke eigenvalues of elliptic cusp forms, these Hecke eigenvalues are always positive. Note that since the Maass space is isomorphic to the space of elliptic cusp forms, multiplicity one theorem holds good for the Hecke eigenforms in this space.

On the contrary, multiplicity one theorem is not known in the orthogonal complement of the Maass space. In this case, along with S. Gun and W. Kohnen [26] we show the existence of infinitely many $n \in \mathbb{N}$ such that $\mu_F(n) \neq \mu_G(n)$ when F and G lie in different eigenspaces. More precisely, we prove the following theorem.

Theorem 0.0.19. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass space and having Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. Also let F and G lie in different eigenspaces. Then for any $\epsilon > 0$, one has*

$$\#\{n \leq x \mid \mu_F(n) \neq \mu_G(n)\} \gg x^{1-\epsilon}$$

where the constant $\gg$ depends on F, G and ϵ.

In particular, the above theorem shows that at least one of F or G has infinitely many non-zero Hecke eigenvalues. This motivated us to the question whether F and G have simultaneous non-zero Hecke eigenvalues. In this context, we have the following theorem;

Theorem 0.0.20. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass space with Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. Also let F and G lie in different eigenspaces. Then for any prime p, there exists an integer n with $1 \leq n \leq 14$ such that*

$$\mu_F(p^n)\mu_G(p^n) \neq 0.$$

We also investigate the question of Hecke eigenvalues which are of different sign. This in turn would ensure simultaneous non-zero Hecke eigenvalues. In this direction we first investigate the question of different signs at primes.

Theorem 0.0.21. *Let $F \in S_{k_1}(\Gamma_2)$ be a Hecke eigenform lying in the orthogonal complement of the Maass space and having Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$. Also assume that there exist $0 < c < 4$ and a Hecke eigenform $G \in S_{k_2}(\Gamma_2)$ lying in the* *orthogonal complement of the Maass space with Hecke eigenvalues* $\{\mu_G(n)\}_{n\in\mathbb{N}}$ *such that*

$$\#\left\{p \le x \mid |\mu_G(p)| > cp^{k_2-\frac{3}{2}}\right\} \;\ge\; \frac{16}{17}\cdot\frac{x}{\log x}$$

for sufficiently large x*. Also assume that* F *and* G *lie in different eigenspaces. Then there exists a set of primes* p *of positive lower density such that* $\mu_F(p)\mu_G(p) \gtrless 0$.

Applying above theorem, we have the following theorem;

Theorem 0.0.22. *Let* $F \in S_{k_1}(\Gamma_2)$ *and* $G \in S_{k_2}(\Gamma_2)$ *be as in Theorem 0.0.21. Then half of the non-zero coefficients of the sequence* $\{\mu_F(n)\mu_G(n)\}_{n\in\mathbb{N}}$ *are positive and half of them are negative.*

Notations

Symbol	**Description**
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The ring of rational integers
$\mathbb{Q}$	The field of rational numbers
$\mathcal{P}$	The set of all rational prime numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
$\mathcal{H}$	The complex upper-half plane
$M_n(\mathbb{Z})$	The set of all $n \times n$ integer matrices.
$M_{n,m}(\mathbb{Z})$	The set of all $n \times m$ integer matrices.
$\mathrm{SL}_2(\mathbb{Z})$	The group of all 2×2 integer matrices with determinant 1.
1_g	The $g \times g$ identity matrix
M^t	The transpose of the matrix M
$Y \geq 0$	The matrix Y is positive semi definite
(a, b)	The greatest common divisor of two natural numbers a and b
$[a, b]$	The least common multiple of two natural numbers a and b
$a \mid b$	a divides b
Γ	The gamma function
$\zeta(s)$	The Riemann zeta function
$\Re(s)$	The real part of a complex number s
$\Im(s)$	The imaginary part of a complex number s

p	prime number
$[x]$	The greatest integer $n \le x$
$\#A$	Number of elements in the set A
$\overline{z}$	Conjugate of the complex number z

We also use the following notations frequently.

1. Let $f, g : \mathbb{R} \to \mathbb{C}$ be functions such that $g(x) > 0$ for all $x \in \mathbb{R}$. We shall say that $f = O(g)$ or $f \ll g$ if there exists a constant $C > 0$ such that

$$|f(x)| \le Cg(x).$$

Further, if there exist constants $C_1, C_2 > 0$ such that

$$C_1 g(x) \le |f(x)| \le C_2 g(x)$$

then we write $f \asymp g$. By $f = o(g)$ and $f(x) \sim g(x)$, we shall denote

$$\lim_{x\to+\infty} \frac{|f(x)|}{g(x)} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \frac{f(x)}{g(x)} = 1$$

respective. We use the symbol $f(x) = \Omega(g(x))$ to indicate that

$$\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} > 0.$$

Whereas, the symbol $f(x) = \Omega_{\pm}(g(x))$ means that

$$\limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0 \quad \text{and} \quad \liminf_{x\to\infty} \frac{f(x)}{g(x)} < 0.$$

2. Let A be a subset of $\mathcal{P}$. We say that the lower natural density of A is greater

than or equal to $\underline{d}(S) \in \mathbb{R}$ if

$$\liminf_{x\to\infty} \frac{\#\{p \in A \mid p \leq x\}}{\#\{p \in \mathcal{P} \mid p \leq x\}} \;\geq\; \underline{d}(S).$$

Further, we shall say that a subset A of $\mathcal{P}$ has natural density $\alpha \in \mathbb{R}$ if

$$\lim_{x\to\infty} \frac{\#\{p \in A \mid p \leq x\}}{\#\{p \in \mathcal{P} \mid p \leq x\}}$$

exists and is equal to α. We shall denote the natural density of $A \subset \mathcal{P}$ by $d(A)$ if it exists.

3. We say that the density of $A \subset \mathbb{N}$ is $d(A)$ if

$$\lim_{x\to\infty} \frac{\#\{n \leq x \mid n \in A\}}{\#\{n \leq x \mid n \in \mathbb{N}\}}$$

exists and is equal to the real number $d(A)$.

Chapter 1

Introduction

"There are five basic operations in arithmetic: addition, subtraction, multiplication, division, and modular forms."

Martin Eichler

In this chapter, we shall give a brief history of the theme around which this thesis is centered. Further, we state our main results and outline the arrangement of the chapters of this thesis.

1.1 History

Modular forms are one of the most fundamental objects in mathematics. They appear in several branches of mathematics such as number theory, arithmetic geometry, representation theory, Riemann surface theory and so on. Thus the theory of modular forms has fascinated mathematicians for a long time.

The theory of modular forms can be traced back to the works of *Jacobi* on elliptic functions. However, the seeds of a comprehensive study of the theory of modular forms were hidden in the seminal paper *"On certain arithmetical functions"* of *Ramanujan* where he introduced the arithmetic function τ as the coefficients of the following formal power series

$$\sum_{n=1}^{\infty} \tau(n) x^n \;:=\; x \prod_{n=1}^{\infty} (1 - x^n)^{24}. \tag{1.1.1}$$

This is now called the *Ramanujan τ-function.* They also appear in the study of the number of representations of an odd integer n as a sum of 24 squares in the following manner

$$r(x_1^2 + \cdots + x_{24}^2, n) \;=\; \frac{16}{691}\sigma_{11}(n) \;+\; \frac{33152}{691}\tau(n),$$

where $r(x_1^2 + \cdots + x_{24}^2, n)$ denotes the number of integral solutions of $x_1^2 + \cdots + x_{24}^2 = n$ and $\sigma_k(n) := \sum_{d|n} d^k$. Ramanujan [79] predicted the following properties of τ-function:

1. $\tau(mn) = \tau(m)\tau(n)$ when $(m, n) = 1$;
2. $\tau(p^{n+2}) = \tau(p)\tau(p^{n+1}) - p^{11}\tau(p^n)$ for any prime p and $n \in \mathbb{N}$;
3. $|\tau(p)| \leq 2p^{11/2}$ for all primes p.

The first two assertions of *Ramanujan* were proved by *Mordell* [62] in 1917. But the third assertion remained unsolved till 1974 when *Deligne* [19] proved it as a consequence of his proof of *Weil's conjectures.*

Hecke, in a series of papers, explained why Ramanujan's conjectures are expected to be true. The theory developed by Hecke is now known as *Hecke theory* for modular forms. In particular, $\tau(n)$ can be interpreted as eigenvalues of certain linear operators which are known as *Hecke operators* acting on the space of modular forms. Unfortunately, this theory could not provide a proof of Ramanujan's third conjecture; it only suggested a general conjecture. For a proof of *Ramanujan's third conjecture*, we had to wait for the subsequent works of *Deligne, Eichler,* *Shimura, Weil* among others which revealed fundamental relations between Hecke theory and algebraic geometry. In particular, the eigenvalues of Hecke operators were interpreted in terms of the zeros of the zeta functions of suitable algebraic varieties over finite fields. This finally led to the proof of *Ramanujan's third conjecture* by *Deligne* [19].

In an attempt to resolve Ramanujan conjecture, *Rankin* and *Selberg* independently developed a theory of L-functions known as *"Rankin-Selberg theory"* of L-functions. In fact, they obtained significant results towards *Ramanujan conjecture.* Around 1950's and 1960's, *Harish-Chandra* and subsequently *Langlands* reformulated the notion of modular forms in the larger framework of representation theory. These works eventually led to the development of *Langlands program* in the theory of automorphic representations.

On the other hand, while studying the analytic theory of quadratic forms, *Siegel* [95, 96] developed the theory of modular forms in several variables which are now called *Siegel modular forms.* In his own words: *"Von der analytischen Theorie der quadratischen Formen her ist man neuerdings zu Funktionen von* $n(n+1)/2$ *Variabeln geführt worden, die für ein beliebiges algebraisches Gebilde vom Geschlecht* n *dasselbe leisten wie die elliptischen Modulfunktionen im Falle* $n = 1$ *und die deshalb Modulfunktionen* n*-ten Grades genannt werden. Diese Funktionen sind von Interesse wegen verschiedenartiger Anwendungen auf Algebra und Arithmetik, und ihre analytischen Eigenschaften lassen sich ziemlich weit verfolgen."* This translates as follows: *From the analytic theory of quadratic forms, we have recently been led to functions of* $n(n+1)/2$ *variables which perform the same for any algebraic structure of genus* n *as the elliptic modular functions in the case* $n = 1$ *and which are therefore called the modular functions of* n*-th degree. These functions are of interest because of various applications to algebra and arithmetic and their analytical properties can be tracked quite widely.* The theory of Siegel modular forms generalizes the theory of elliptic modular forms.

Soon after the work of *Hecke*, attempts were made to develop the theory of Hecke operators for Siegel modular forms. Hecke theory provided a framework to interlink the theory of Siegel modular forms with algebraic geometry, representation theory and Galois theory. The theory of Siegel modular forms was further developed by *Andrianov, Eichler, Kohnen, Maass, Shimura, Zagier* and others. Still in the case of degree $n > 1$, the study of the theory of Siegel modular forms is far from being complete.

The main arithmetic application of modular forms had been the analytical theory of integral quadratic forms till mid twentieth century when *Shimura* and *Taniyama* proposed the famous modularity conjecture relating modular forms of weight 2 to elliptic curves over $\mathbb{Q}$. The *Taniyama-Shimura conjecture* predicts that the zeta function associated to an elliptic curve over $\mathbb{Q}$ can be realized as the zeta function of a cusp form. In 1985, *Frey* made the remarkable observation that the conjecture of *Taniyama-Shimura* would imply *Fermat's last theorem.* The precise relations among these two were formulated by *Serre* [93] and established later by *Ribet*, which allowed *Wiles* in 1995 to prove *Fermat's last theorem.* This is one of the biggest achievements of mathematics.

One would expect that the relation between zeta functions of elliptic curves and zeta functions of elliptic modular forms described by *Taniyama-Shimura conjecture* is only a particular case of some general relations between zeta functions of algebraic varieties and zeta functions of automorphic forms. It is believed that zeta functions of abelian varieties should be related to zeta functions of Siegel modular forms. Thus the theory of Hecke operators and L-functions of the Siegel modular forms are an integral part of the theory of Siegel modular forms. All these make the theory rich and motivate us to investigate the following question around which this thesis is centered.

Question 1. *Given two Siegel modular forms F and G of weights k_1 and k_2 respectively, how do we determine whether $F = G$?*

For elliptic modular forms which are Siegel modular forms of degree one, the above question has been addressed by several authors, namely Kohnen, Sengupta [39], Kowalski, Lau, Soundararajan, Wu [41], Luo [52], Ramakrishnan [51], Matomäki [57], R. Murty [64] among others. In this doctoral thesis, we address Question 1 by appealing to the arithmetic properties of the Hecke eigenvalues of Siegel cusp forms of degree one and two. In particular, our focus is to exploit arithmetic properties of these Hecke eigenvalues as essential tools to distinguish cusp forms of degree one and degree two.

1.2 Main results

For integers $k, g \geq 1$, let $\Gamma_g := \mathrm{Sp}_g(\mathbb{Z})$ be the Siegel modular group of degree g and $S_k(\Gamma_g)$ be the space of cuspidal Siegel modular forms of weight k and degree g for Γ_g. For a positive integer n, recall that the n-th Hecke operator $T_g(n)$ on the space $S_k(\Gamma_g)$ is given by

$$T_g(n)F := n^{gk-\frac{g(g+1)}{2}} \sum_{\gamma \in \Gamma_g \backslash \mathcal{O}_{g,n}} F \mid \gamma,$$

where

$$\mathcal{O}_{g,n} := \left\{ \gamma \in M_{2g}(\mathbb{Z}) \mid \gamma^t J_g \gamma = n J_g \right\}, \quad J_g := \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

and

$$F \mid \gamma \ := \ \det\, (CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1}), \quad \gamma := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

It is known that the complex vector space $S_k(\Gamma_g)$ has a basis consisting of eigenvectors of all the Hecke operators $T_g(n)$. Let $F \in S_k(\Gamma_g)$ be such an eigenvector of $T_g(n)$ with eigenvalue $\mu_F(n)$, that is, $T_g(n)F = \mu_F(n)F$ for all $n \in \mathbb{N}$. Then one knows that μ_F is a multiplicative function.

Note that for degree $g = 1$, the space $S_k(\Gamma_1)$ is nothing but the space of elliptic cusp forms of level 1 and weight k. In this case, by a celebrated work of Deligne, one knows that the Ramanujan-Petersson conjecture is true, that is, for any prime p, one has

$$|\mu_F(p)| \ \leq \ 2p^{(k-1)/2}.$$

It is natural to ask whether this upper-bound is optimal. One way to answer this question is to derive an omega result. In 1983, R. Murty [64] proved the following theorem.

Theorem 1.2.1. *[R. Murty] Let $f \in S_k(\Gamma_1)$ be a normalized Hecke eigenform. Then there exists a constant $c > 0$ such that*

$$\mu_F(n) \ = \ \Omega_{\pm}\left(n^{(k-1)/2} \exp\left(\frac{c \log n}{\log \log n}\right)\right).$$

Here for any arithmetic functions f and g with $g(n) > 0$ for all $n \in \mathbb{N}$, the symbol $f(n) = \Omega_{\pm}(g(n))$ means

$$\limsup_{n\to\infty} \frac{f(n)}{g(n)} \ > \ 0 \quad \text{and} \quad \liminf_{n\to\infty} \frac{f(n)}{g(n)} \ < \ 0.$$

The above result shows that the sequence $\{\mu_F(n)\}_{n\in\mathbb{N}}$ changes sign infinitely often. The question of sign changes of Hecke eigenvalues of modular forms has been addressed by many mathematicians (for example see [13], [27], [36], [38], [41], [57], [59]). The recent works of Kowalski, Lau, Soundararajan and Wu [41] and of Matomäki [57] prove that any normalized Hecke eigenform $f \in S_k^{new}(N)$ is uniquely determined by the signs of its Hecke eigenvalues at primes. (See Kohnen and Sengupta [39], Gun, Kohnen and Rath [27] and Kumari and R. Murty [45] for the analogous results for arbitrary cusp form $f \in S_k(N)$ in terms of the signs of Fourier coefficients.) Here $S_k^{new}(N)$ denotes the space of newforms of weight k for $\Gamma_0(N)$ (see section 2.1.3 for a precise definition). More precisely, for $z \in \mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$, $q := e^{2\pi i z}$, let

$$(1.2.1) \qquad f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_{k_1}^{new}(N_1) \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} a_g(n) q^n \in S_{k_2}^{new}(N_2)$$

be normalized Hecke eigenforms. In this case, the normalization ensures that

$$\mu_f(n) = a_f(n) \quad \text{and} \quad \mu_g(n) = a_g(n)$$

for all $(n, N_1 N_2) = 1$. To state the result of Kowalski, Lau, Soundararajan and Wu, we need the following definitions of analytic density of a subset of the set $\mathcal{P}$ of primes and of CM forms.

Definition 1.2.2. *A subset E of the set $\mathcal{P}$ of primes has analytic density $\kappa > 0$ if*

$$\lim_{\sigma \to 1+} \frac{\sum_{p \in E} 1/p^{\sigma}}{\sum_{p \in \mathcal{P}} 1/p^{\sigma}} = \kappa.$$

Definition 1.2.3. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform.*

- *We say that f has complex multiplication (or of CM type) if there exists a non-trivial Dirichlet character χ modulo D such that*

  $$a_f(p)\chi(p) = a_f(p)$$

  for all but finitely many primes p.

- *A form is called a non-CM form or of non-CM type if it is not of CM type.*

If $f \in S_k^{new}(N)$ is a CM form and χ is as in the above definition, then $a_f(p) = 0$ for all primes p such that $\chi(p) = -1$. Further, by [83, Corollary 3.10], we know that there are no CM forms of level N if N is square free.

With these notations in place, we can state the theorem of Kowalski, Lau, Soundararajan and Wu [41, Theorem 4].

Theorem 1.2.4. *[Kowalski, Lau, Soundararajan and Wu] Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms with Fourier expansions as in* (1.2.1)*.*

1. *If $a_f(p)$ and $a_g(p)$ have same sign for every prime p except those in a set E of analytic density zero, then $f = g$.*

2. *Also assume that both f and g are non-CM forms. If $a_f(p)$ and $a_g(p)$ have same sign for every prime p except those in a set E of analytic density $\leq 1/32$, then $f = g$.*

Shortly after this result, Matomäki [57, Theorem 2] strengthen the second part of above theorem by showing the following.

Theorem 1.2.5. *[Matomäki] Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms with Fourier expansions as in* (1.2.1)*. Also assume that both f and g are non-CM forms. If $a_f(p)$ and $a_g(p)$ have same sign for every prime p except those in a set E of analytic density $\leq 6/25$, then $f = g$.*

We now briefly describe our results in this context. In [28], we investigate simultaneous sign change and non-vanishing of Hecke eigenvalues of newforms which are normalized Hecke eigenforms. In particular, we have the following theorem.

Theorem 1.2.6. *Let N_1, N_2 be square-free, $N := lcm[N_1, N_2]$ and $f \in S_{k_1}^{new}(N_1)$, $g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with Fourier expansions as* *in* (1.2.1)*. Then there exists a prime power p^α, $\alpha \le 2$ with*

$$p^\alpha \ll_\epsilon \max\left\{\exp\left(c\log^2(\sqrt{\mathfrak{q}(f)}+\sqrt{\mathfrak{q}(g)})\right), \left[N^2\left(1+\frac{|k_2-k_1|}{2}\right)\left(\frac{k_1+k_2}{2}\right)\right]^{1+\epsilon}\right\}$$

such that $a_f(p^\alpha)a_g(p^\alpha) < 0$. Here $c > 0$ is an absolute constant and $\mathfrak{q}(f)$, $\mathfrak{q}(g)$ are analytic conductors of the Rankin-Selberg L-functions of f and g respectively. Note that

$$\mathfrak{q}(f) \ll k_1^2 N_1^2 \log\log N_1 \quad \textit{and} \quad \mathfrak{q}(f) \ll k_2^2 N_2^2 \log\log N_2.$$

This can be thought of as sign change analogue of the classical Sturm's bound. Ghosh and Sarnak [24], in their study of distribution of real zeros of Hecke eigenforms, relate the question of sign changes of Fourier coefficients of Hecke eigenforms to the question of distribution of real zeros of those forms. (Also see the recent paper of Matomäki [58] in this context.) In a recent work [30] with Gun, we relate the question of simultaneous sign changes of Fourier coefficients of primitive cusp forms to multiplicity one theorem for those forms. More precisely, for any $f \in S_k(N)$ which is a normalized Hecke eigenform with Fourier coefficients $a_f(n)$, let us set

$$\lambda_f(n) := \frac{a_f(n)}{n^{(k-1)/2}}.$$

With this notation in place, we show the following.

Theorem 1.2.7. *Let $f \in S_{k_1}(N_1)$ and $g \in S_{k_2}(N_2)$ be normalized Hecke eigenforms and p be a prime such that $(p, N_1N_2) = 1$. Then the following conditions are equivalent;*

1. *there exist infinitely many $m \ge 1$ such that $\lambda_f(p^m)\lambda_g(p^m) > 0$ and infinitely many $m \ge 1$ such that $\lambda_f(p^m)\lambda_g(p^m) < 0$;*

2. *one has $\lambda_f(p) \ne \lambda_g(p)$.*

This theorem allows us to estimate the density of the set of primes p for which the sequence $\{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ changes sign infinitely often. In order to state our next theorem, we shall need the following notion of natural density of a subset of the set of primes.

Definition 1.2.8. *Let A be a subset of the set of primes $\mathcal{P}$. We say that the natural density of the set A is $d(A)$ if the limit*

$$\lim_{x\to\infty} \frac{\#\{p \leq x \mid p \in A\}}{\#\{p \leq x \mid p \in \mathcal{P}\}}$$

exists and is equal to $d(A)$.

Let $f \in S_k^{new}(N)$ be a CM form. Then by a work of Ribet [82], one knows that there exists a Hecke character χ of an imaginary quadratic field K such that the Fourier coefficients of f are determined by χ. In this case, we shall say that the form f has CM by the imaginary quadratic field K. Now by applying Theorem 1.2.7, we have

Theorem 1.2.9. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms with Fourier expansions as in* (1.2.1) *and S be the set of primes p for which the sets*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) > 0\} \quad \text{and} \quad \{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) < 0\}$$

are infinite. Then,

1. *if at least one of f or g is a non-CM form, then*
   - *the natural density of S is 1 provided $f \neq g \otimes \chi$ for any Dirichlet character χ;*
   - *the natural density of S is $1/2$ if $f = g \otimes \chi$ for some Dirichlet character χ.*

2. if both f and g are of CM type, then

- *the lower natural density of S is greater than or equal to $1/2$ if either $k_1 \neq k_2$ or f and g have CM by different quadratic fields;*
- *the lower natural density of S is greater than or equal to $1/8$ if $k_1 = k_2$ and the forms f and g have CM by the same field.*

Theorem 1.2.9 improves the following result of Gun, Kohnen and Rath [27, Theorem 3].

Theorem 1.2.10. *[Gun, Kohnen and Rath] Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms with Fourier expansions as in (1.2.1). Then there exists an infinite set S of primes p such that the sets*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) > 0\} \quad \text{and} \quad \{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) < 0\}$$

are infinite.

If we assume that at least one of f or g is a non-CM form, then we can prove the following stronger result.

Theorem 1.2.11. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms with Fourier expansions as in (1.2.1) and not both of CM type. For any positive integer j, let S_j be the set of primes p such that*

$$\left\{m \in \mathbb{N} \mid a_f(p^{jm})a_g(p^{jm}) > 0\right\} \quad \text{and} \quad \left\{m \in \mathbb{N} \mid a_f(p^{jm})a_g(p^{jm}) < 0\right\}$$

are infinite. Then,

1. if $f \neq g \otimes \chi$ for any Dirichlet character χ, then the natural density of S_j is equal to one for any $j \in \mathbb{N}$.

2. *when $f = g \otimes \chi$ for some Dirichlet character χ, then*

   - *if j is odd, then the natural density of S_j is equal to $1/2$;*
   - *if j is even, then the natural density of S_j is equal to zero.*

The above theorem can be thought of as a generalization of the following result of Kohnen and Martin [40].

Theorem 1.2.12. [Kohnen and Martin] *Let $f \in S_k(1)$ be a normalized Hecke eigenform. Then for any integer $j \geq 1$ and for almost all primes p, the sequence $\{a_f(p^{nj})\}_{n\in\mathbb{N}}$ changes sign infinitely often.*

As mentioned in Remark 3.1 of [17], their proof does not work for natural numbers j that are divisible by 4. In [30], along with Gun we proved the following theorem.

Theorem 1.2.13. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform and $j \geq 1$ be a natural number. Consider the set S_j of primes p for which the sets*

$$\left\{m \in \mathbb{N} \mid a_f(p^{jm}) > 0\right\} \quad \text{and} \quad \left\{m \in \mathbb{N} \mid a_f(p^{jm}) < 0\right\}$$

are infinite. Then,

1. *if f is a non-CM form, then the natural density of S_j is 1;*
2. *if f is of CM type and*
   - *$4|j$, then the natural density of S_j is $1/2$;*
   - *$4 \nmid j$, then the natural density of S_j is 1.*

The following theorem leads to the proof of the previous one.

Theorem 1.2.14. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform and j be a positive integer. Then for almost all primes p, the following conditions are equivalent;*

1. *there exists infinitely many natural numbers $m \geq 1$ such that $\lambda_f(p^{jm}) > 0$ and infinitely many natural numbers $m \geq 1$ such that $\lambda_f(p^{jm}) < 0$;*

2. *one has*
$$\lambda_f(p) \notin \begin{cases} \{2\} & \text{for } j \text{ is odd}; \\ \{2, -2\} & \text{for } j \equiv 2 \ (mod\ 4); \\ \{-2, 0, 2\} & \text{for } j \equiv 0 \ (mod\ 4). \end{cases}$$

Further, when $k \geq 4$ or $j = 1$, then the above equivalence is true for all primes p with $(p, N) = 1$.

We now address the question of non-vanishing of Hecke eigenvalues of newforms. The study of non-vanishing of Hecke eigenvalues of newforms is inspired by the folklore conjecture of Lehmer [49] which predicts that $\tau(n) \neq 0$ for all $n \in \mathbb{N}$. Here τ is the Ramanujan τ-function defined by the formal identity (1.1.1). It is well known that the formal sum in (1.1.1) determines unique normalized Hecke eigenform of weight 12 of degree one for the full modular group. One of the most notable results in this direction is due to Serre [91, 92] which states that the set of primes p such that $\tau(p) = 0$ has natural density zero. In fact, his result (see also [91]) characterizes non-CM forms as follows: a Hecke eigen newform is non-CM if and only if the set of primes p for which the p-th Hecke eigenvalue vanishes has natural density zero.

While the above results on sign changes of Hecke eigenvalues imply non-vanishing of the same, one can prove stronger results in this direction. In a joint work with Gun and Kumar [28], we investigate the non-vanishing nature of the sequence $\{a_f(p^m)a_g(p^m)\}_{m \in \mathbb{N}}$. Our first result in this set-up is the following.

Theorem 1.2.15. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with Fourier expansion as in* (1.2.1)*. Then for all primes p with $(p, N_1N_2) = 1$, the set*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0\}$$

has positive density.

We now state our next theorem which strengthens a recent result (namely Theorem 1.2) of Kumari and R. Murty [45].

Theorem 1.2.16. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized non-CM Hecke eigenforms with Fourier expansions as in* (1.2.1)*. Then there exists a set S of primes with natural density one such that for any $p \in S$ and integers $m, m' \geq 1$, we have*

$$a_f(p^m)a_g(p^{m'}) \neq 0.$$

Now we address the question of first simultaneous non-vanishing, analogous to that considered in Theorem 1.2.6. In particular, we have the following theorem.

Theorem 1.2.17. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with Fourier expansions as in* (1.2.1)*. Also assume that $N :=$ lcm $[N_1, N_2] > 12$. Then there exists a positive integer $1 < n \leq (2\log N)^4$ with $(n, N) = 1$ such that*

$$a_f(n)a_g(n) \neq 0.$$

Further, when N is odd, then there exists an integer $1 < n \leq 16$ with $(n, N) = 1$ such that

$$a_f(n)a_g(n) \neq 0.$$

Note that while $a_f(1)a_g(1) = 1$, our goal is to find the first $n > 1$ with $(n, N) = 1$ for which $a_f(n)a_g(n) \neq 0$; in other words to determine the first non-trivial simultaneous non-vanishing.

Now we investigate similar questions for higher degree Siegel cusp forms. In this direction the generalized Ramanujan-Petersson conjecture, as formulated in [73] implies that for any prime p and $\epsilon > 0$, one has

$$|\mu_F(p)| \ll_{g,\epsilon} p^{gk/2-g(g+1)/4+\epsilon}. \tag{1.2.2}$$

However when $g = 2$, this is known to hold for all Hecke eigenforms except for those lying in the Maass subspace S_k^* of $S_k(\Gamma_2)$. In fact, the Hecke eigenforms in S_k^* do not satisfy the Ramanujan-Petersson bound (1.2.2). This is a deep result due to Weissauer [99].

The above result motivates us to study the arithmetic properties such as bounds, growth, non-vanishing nature and distribution of Hecke eigenvalues of the eigenforms which are in the Maass subspace S_k^* of $S_k(\Gamma_2)$ and hence inaccessible vis-a-vis Ramanujan-Petersson bounds.

In a joint work with Gun and Sengupta [31], we proved the following theorems.

Theorem 1.2.18. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there exists an absolute constant $c > 0$ such that*

$$\mu_F(n) = \Omega\left(n^{k-1} exp\left(c\frac{\sqrt{\log n}}{\log\log n}\right)\right).$$

This gives an improvement of the following result of Das and Sengupta [16].

Theorem 1.2.19. *[Das and Sengupta] Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then*

$$\mu_F(n) = \Omega\left(n^{k-1}\frac{\sqrt{\log n}}{\log\log n}\right).$$

We next show that the omega result proved in Theorem 1.2.18 is not too far from an upper bound one can derive. In particular, we have the following theorem.

Theorem 1.2.20. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there exists an absolute constant $c_1 > 0$ such that*

$$\mu_F(n) \ \leq \ n^{k-1} exp\left(c_1 \sqrt{\frac{\log n}{\log\log n}}\right)$$

for all $n \in \mathbb{N}$ with $n \geq 3$.

Theorem 1.2.20 improves an earlier result of Pitale and Schmidt (see page 101 of [75]). We also prove the following lower bound.

Theorem 1.2.21. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there exist absolute constants $c_2, c_3 > 0$ such that*

$$\mu_F(n) \geq c_2 n^{k-1} exp\left(-c_3 \sqrt{\frac{\log n}{\log\log n}}\right)$$

for all positive integers $n \geq 3$.

As a corollary, we derive the following result of Breulmann [12] whose proof is rather different from ours.

Corollary 1.2.22. *If $F \in S_k^*$ is a non-zero Hecke eigenform with Hecke eigenvalues $\mu_F(n)$, then $\mu_F(n) > 0$.*

Note that $\mu_F(n)/n^{k-1} > 0$. One might wonder whether it is possible to improve the above lower bound, that is, whether there exists a real number $c > 0$ such that $\mu_F(n)/n^{k-1} > c$ for all $n \in \mathbb{N}$. Our next theorem precludes such a possibility.

Theorem 1.2.23. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then we have*

$$\liminf_{n\to\infty} \frac{\mu_F(n)}{n^{k-1}} = 0.$$

In particular, zero is a limit point of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$. This motivated us to investigate the set of limit points of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$. In this direction, we have the following result.

Theorem 1.2.24. *Let $F \in S_k^*$ be a non-zero Hecke eigenform. Then there are infinitely many limit points of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$ in $(1,\infty)$ and infinitely many limit points in $(0,1)$.*

This summarizes our study of arithmetic properties of Hecke eigenvalues of Hecke eigenforms lying in the Maass subspace S_k^*. Unlike the Hecke eigenvalues of elliptic Hecke eigenforms, these Hecke eigenvalues are always positive. Note that the Maass subspace is isomorphic to the space of elliptic cusp forms by the *Saito-Kurokawa* lift which was constructed in a series of papers by Maass [54, 55, 56], Andrianov [4] and Zagier [103] (also see [20]). Hence multiplicity one theorem holds good for the Hecke eigenforms lying in this subspace.

On the contrary, multiplicity one theorem is not known for the Hecke eigenforms in the orthogonal complement of the Maass subspace with respect to Petersson inner product (see section 2.2.1 for definition). In fact one has the following conjecture (see Saha [86]).

Conjecture 1.2.25. *Let F and G be two Hecke eigenforms in $S_k(\Gamma_2)$ such that for all primes p, we have an equality of Hecke eigenvalues $\mu_F(p) = \mu_G(p)$ and $\mu_F(p^2) = \mu_G(p^2)$. Then there exists a constant c such that $F = cG$.*

By a recent work of Saha [86], one knows that the above conjecture follows from a version of generalised Böcherer's conjecture (see [86, 11] for further details). In this context, along with Gun and Kohnen [26] we show the existence of infinitely many $n \in \mathbb{N}$ such that $\mu_F(n) \neq \mu_G(n)$ when F and G lie in different eigenspaces. More precisely, we prove the following theorem.

Theorem 1.2.26. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace and having Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. Also let F and G lie in different* *eigenspaces. Then for any $\epsilon > 0$, one has*

$$\#\left\{n \leq x \mid \mu_F(n) \neq \mu_G(n)\right\} \gg x^{1-\epsilon}$$

where the constant $\gg$ depends on F, G and ϵ.

In particular, the above theorem shows that at least one of F or G has infinitely many non-zero Hecke eigenvalues. This motivated us to investigate the question whether an Hecke eigenform F has infinitely many non-zero eigenvalues. In fact, we investigate whether F and G have infinitely many simultaneous non-zero Hecke eigenvalues. In this context, we have the following theorem;

Theorem 1.2.27. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace with Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. Also let F and G lie in different eigenspaces. Then for any prime p, there exists an integer n with $1 \leq n \leq 14$ such that*

$$\mu_F(p^n)\mu_G(p^n) \neq 0.$$

We also investigate the question of Hecke eigenvalues which are of different sign. This in turn would ensure simultaneous non-zero Hecke eigenvalues. In this direction we first investigate the question of different signs at primes.

Theorem 1.2.28. *Let $F \in S_{k_1}(\Gamma_2)$ be a Hecke eigenform lying in the orthogonal complement of the Maass subspace and having Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$. Also assume that there exist $0 < c < 4$ and a Hecke eigenform $G \in S_{k_2}(\Gamma_2)$ lying in the orthogonal complement of the Maass subspace with Hecke eigenvalues $\{\mu_G(n)\}_{n\in\mathbb{N}}$ such that*

$$\#\left\{p \leq x \mid |\mu_G(p)| > cp^{k_2-\frac{3}{2}}\right\} \ \geq \ \frac{16}{17} \cdot \frac{x}{\log x}$$

for sufficiently large x. Also assume that F and G lie in different eigenspaces. Then *there exists a set of primes p of positive lower density such that $\mu_F(p)\mu_G(p) \gtrless 0$.*

Applying above theorem, we have the following theorem;

Theorem 1.2.29. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Theorem 1.2.28. Then half of the non-zero coefficients of the sequence $\{\mu_F(n)\mu_G(n)\}_{n\in\mathbb{N}}$ are positive and half of them are negative.*

1.3 Arrangement of the Thesis

The thesis is organized as follows. In the next chapter, we shall list some basic definitions and results from various branches of number theory. We shall use these notions and results in the subsequent chapters of this thesis. The third and fourth chapters are devoted to the study of arithmetic properties of Hecke eigenvalues of elliptic cuspforms.

More precisely in chapter 3, we shall investigate sign changes of Hecke eigenvalues of elliptic modular forms in details. Our results in this context suggest some approach to determine normalized Hecke eigenforms lying in newform subspace uniquely. We shall first address a question which can be thought of as a variant of classical Sturm's theorem. Next we consider simultaneous sign changes in short intervals. Finally, we establish a relation between simultaneous sign change and multiplicity one theorem which allows us to derive some quantitative sign change results. Some of the main ingredients to prove these results are Hecke relation, Deligne's bound, Rankin-Selberg method, Sato-Tate conjecture / theorem, joint Sato-Tate distribution and properties of Hecke characters.

In Chapter 4, we shall study simultaneous non-vanishing of Hecke eigenvalues of elliptic modular forms. We also study non-vanishing of the coefficients of symmetric power L-functions attached to a normalized Hecke eigenforms lying in the newforms space. Some important ingredients to prove these theorems are Deligne's bound, Hecke relation and properties of $\mathfrak{B}$-free numbers.

In the last two chapters, we shall investigate arithmetic properties of Hecke eigenvalues of Siegel modular forms of degree two. In the penultimate chapter, we shall discuss our results on Hecke eigenvalues of cuspidal Siegel modular forms of degree two which lie in the *Maass subspace*. For these eigenvalues, we shall investigate bounds, omega results, existence and distribution of limit points of these eigenvalues. One of the main ingredients to prove these theorems is the *Saito-Kurokawa lift*.

In the last chapter of the thesis, we shall investigate arithmetic properties like multiplicity one, non-vanishing, sign changes of Hecke eigenvalues of Siegel cuspforms which do not lie in the *Maass subspace*. Some of the results in this chapter are under some mild conditions. Important ingredients to prove these theorems are Hecke relation and the analytic properties of the *Rankin-Selberg L-function* attached to Siegel cuspforms which are Hecke eigenforms.

Chapter 2

Preliminaries

In this chapter, we list some basic results which are required for this thesis. To keep the exposition simple, we shall discuss elliptic modular forms and Siegel modular forms of degree two separately. In the first section, we discuss necessary definitions and results required from the theory of elliptic modular forms. The second section is devoted to the definitions and results from the theory of Siegel modular forms of degree two. For the exposition of these two sections we mainly rely on Andrianov [3], Cohen and Stromberg [14] and Shimura [94]. In the last section, we shall discuss some properties of $\mathfrak{B}$-free numbers.

2.1 Modular forms

We start this section by recalling the definition of elliptic modular forms. Then we introduce the concepts of Petersson inner product, Hecke operators, theory of oldforms and newforms. For the results quoted in the last topic we follow Atkin and Lehner [6] (also see Li [50]). Finally, we discuss analytic properties of various L-functions attached to newforms. Let us start with the following notations.

Let $\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ be the complex upper-half plane and

$$\mathrm{SL}_2(\mathbb{Z}) \;:=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; ad - bc = 1 \right\}.$$

For any positive integer N, let $\Gamma_0(N)$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined as follows:

$$\Gamma_0(N) \;:=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; N \mid c \right\}.$$

Note that $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$. The group $\Gamma_0(N)$ acts on $\mathcal{H}$ as follows:

(2.1.1)
$$\begin{aligned} \Gamma_0(N) \times \mathcal{H} &\rightarrow \mathcal{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) &\mapsto \frac{az+b}{cz+d}. \end{aligned}$$

With these notations in place, we are now ready to define elliptic modular forms.

Definition 2.1.1. *A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a modular form of weight k ($k \in \mathbb{Z}$) and for the group $\Gamma_0(N)$ if it satisfies the following properties:*

1. *for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $z \in \mathcal{H}$, we have*

$$f(\gamma z) \;=\; (cz+d)^k f(z).$$

2. *for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, the function $(cz+d)^{-k} f(\gamma z)$ is bounded in any domain of the form $\{z \in \mathbb{C} \mid \Im(z) > c_1\}$ for any $c_1 > 0$.*

Let $M_k(N)$ be the set of modular forms of weight k and of level N, that is, for the group $\Gamma_0(N)$. One can show that $M_k(N)$ is a finite dimensional complex vector space. For any $f \in M_k(N)$, we have its Fourier expansion as follows:

$$f(z) \;=\; \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q := e^{2\pi i z}$. Then the space of cusp forms which will be denoted by $S_k(N)$ is defined by

$$S_k(N) \;:=\; \left\{ f \in M_k(N) \;\middle|\; a_f(0) = 0 \right\}.$$

Note that $S_k(N)$ is a vector subspace of $M_k(N)$. Rest of the section is devoted to find a basis of this space which is good for our purpose. We also discuss some of the properties of this basis.

2.1.1 Petersson inner product

Recall that the action of the group $\Gamma_0(N)$ on $\mathcal{H}$ is defined by (2.1.1). Note that the measure

$$d\mu(z) \;:=\; \frac{dxdy}{y^2},$$

where $x := \Re(z)$ and $y := \Im(z)$ on $\mathcal{H}$ is invariant under this action. That is, for any Borel set A of $\mathcal{H}$, we have $d\mu(\alpha A) = d\mu(A)$ for $\alpha \in \mathrm{SL}_2(\mathbb{Z})$. Thus for any $f, g \in M_k(N)$, we see that

$$f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

is $\Gamma_0(N)$-invariant. Here $\overline{g(z)}$ denotes the complex conjugate of $g(z)$. Therefore we can define

$$\langle f, g \rangle \;:=\; \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathcal{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2} \tag{2.1.2}$$

provided the integral converges absolutely. Since a cusp form f has exponential decay with respect to y, the above integral converges absolutely if at least one of

f or g is a cusp form. Observe that the integral defined by (2.1.2) gives an inner product on the space $S_k(N)$. This inner product is called the *Petersson inner product*. Thus $S_k(N)$ is a finite dimensional Hilbert space with respect to this inner product.

2.1.2 Hecke operators

We now define Hecke operators on the space of modular forms $M_k(N)$. There are many approaches to Hecke operators. Here we follow the book of Cohen and Stromberg [14]. For positive integers m and N, consider the set

$$\mathcal{O}_m(N) := \left\{ A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; (a, N) = 1,\ N \mid c,\ \det A = m \right\}.$$

Note that the group $\Gamma_0(N)$ acts on the set $\mathcal{O}_m(N)$ by left multiplication of matrices.

Definition 2.1.2. *Let $f \in M_k(N)$. For $(m, N) = 1$, the m-th Hecke operator $T(m)$ on the space $M_k(N)$ is defined by*

$$T(m)f := m^{k-1} \sum_{\gamma \in \Gamma_0(N) \backslash \mathcal{O}_m(N)} f \mid \gamma,$$

where $(f \mid \gamma)(z) := (cz + d)^{-k} f(\gamma z)$ *if* $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The spaces $M_k(N)$ and $S_k(N)$ behaves well under the action of these operators. More precisely, the following theorem holds.

Theorem 2.1.3. *The following statements are true.*

1. *The spaces $M_k(N)$ and $S_k(N)$ are stable under the action of Hecke operators $T(m)$ for all $(m, N) = 1$.*

2. *For positive integers* $(n, N) = 1 = (m, N)$*, we have* $T(m)T(n) = T(n)T(m)$*. Further, if* $(n, m) = 1$ *then* $T(mn) = T(m)T(n)$*. Moreover, for any prime* $(p, N) = 1$ *we have* $T(p^{n+1}) = T(p)T(p^n) - p^{k-1}T(p^{n-1})$ *for any* $n \in \mathbb{N}$*.*

3. *For any* $(m, N) = 1$*, the Hecke operator* $T(m)$ *is a Hermitian operator on the space* $S_k(N)$ *with respect to the* Petersson inner product*. That is, for any* $f, g \in S_k(N)$ *and* $(m, N) = 1$*, we have* $\langle T(m)f, g\rangle = \langle f, T(m)g\rangle$*.*

Before proceeding further let us recall the following theorem from linear algebra.

Theorem 2.1.4. *Let* V *be a finite dimensional vector space over a field* F *and* $\{A_\alpha\}_\alpha$ *be a family of diagonalizable operators on* V *such that they commute with each other. Then there exists a basis of* V *consisting of eigenvectors of all* A_α*.*

Proof. Since V is a finite dimensional vector space, the space of linear operators on V is also finite dimensional vector space over F. Let $\{A_1, A_2, \cdots, A_r\}$ be a generating set of the subspace generated by $\{A_\alpha\}_\alpha$. First note that it is sufficient to prove the statement for the set $\{A_i\}_{1\le i\le r}$. We shall use induction on r to prove the statement.

For $r = 1$, it is clear. For $r \ge 2$, let λ be an eigenvalue of A_r with eigenspace E_λ, that is,

$$E_\lambda \ := \ \big\{v \in V \mid A_r v = \lambda v\big\}.$$

By commutativity, we have $A_r(A_i v) = A_i(A_r v) = \lambda A_i v$ for all $v \in E_\lambda$. This shows that $A_i E_\lambda \subset E_\lambda$. Now consider the set of operators $\{A_i|_{E_\lambda} \mid 1 \le i \le r-1\}$. Observe that the operators in this set are diagonalizable and commute with each other. Hence by induction hypothesis, there is a basis of E_λ consisting of eigenvectors of all $\{A_i|_{E_\lambda} \mid 1 \le i \le r\}$ as every vector in E_λ is an eigenvector of A_r. Note that the space V can be written as

$$V = \bigoplus_\lambda E_\lambda,$$

where λ varies over the set of all distinct eigenvalues of A_r and each E_λ has a basis consisting of eigenvectors of all operators $\{A_i|_{E_\lambda} \mid 1 \le i \le r\}$. Hence the space V has a basis consisting of eigenvectors of all operators $\{A_i \mid 1 \le i \le r\}$. This completes the proof. $\square$

Remark 2.1.5. *In fact, if V is a finite dimensional Hilbert space, then one can show that there exists an orthogonal basis of V consisting of eigenvectors of all operators $\{A_\alpha\}_\alpha$.*

We also need spectral theorem from linear algebra.

Theorem 2.1.6. *Let V be a finite dimensional real or complex vector space which is an inner product space. Also let A be a Hermitian operator on V. Then there exists an orthogonal basis of V consisting of eigenvectors of A.*

Thus if A is a Hermitian operator on an inner product space V, then one can write

$$V = \bigoplus_\lambda V_\lambda.$$

Here the summation runs over the distinct eigenvalues of A and eigenspace V_λ is defined by

$$V_\lambda := \big\{v \in V \mid Av = \lambda v\big\}.$$

For a proof of the spectral theorem, one can see Lang [47, page 268].

As an immediate consequence of Theorem 2.1.3, Theorem 2.1.4 and Theorem 2.1.6, one can derive the following important theorem.

Theorem 2.1.7. *The space of cusp forms $S_k(N)$ has an orthogonal basis consisting of eigenfunctions of $T(m)$ for all $(m, N) = 1$.*

Definition 2.1.8. *A cusp form $f \in S_k(N)$ is said to be a Hecke eigenform if it is an eigenfunction for each $T(m)$ with $(m, N) = 1$.*

Note that the basis in Theorem 2.1.7 need not be unique. In the next subsection, we shall discuss how to find a natural basis for the space $S_k(N)$.

2.1.3 Oldforms and Newforms

In this subsection, we shall discuss newform theory developed by Atkin and Lehner [6]. Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k(N)$ be a Hecke eigenform with Hecke eigenvalues $\{\lambda(m)\}_{(m,N)=1}$, that is, for any $(m, N) = 1$, we have $T(m)f = \lambda(m)f$. Then by Theorem 2.1.3, for any $(m, N) = 1$, we have

$$\lambda(m)a_f(n) \;=\; \sum_{d|(m,n)} d^{k-1}a_f(nm/d^2).$$

Thus for $n = 1$, we have $\lambda(m)a_f(1) = a_f(m)$ for any $(m, N) = 1$. From this, we can not deduce that $a_f(1) \neq 0$. In fact, there are eigenforms $f \neq 0$ such that $a_f(1) = 0$. These are the forms which come from lower levels. More precisely, for any positive integer $M \mid N$ and $f \in S_k(M)$, we see that

$$f \in S_k(N) \quad \text{and} \quad f(dz) \in S_k(N) \quad \text{for any} \quad d \mid (N/M).$$

Thus for any $d \mid (N/M)$, we have a well defined map

$$\begin{aligned} B_d : S_k(M) &\to S_k(N) \\ f(z) &\mapsto f(dz) \end{aligned}$$

Now we define the subspace of oldforms $S_k^{old}(N)$ of $S_k(N)$ as follows:

$$S_k^{old}(N) \;:=\; \sum_{\substack{dM \,\mid\, N, \\ M \neq N}} B_d(S_k(M)).$$

Definition 2.1.9. *The subspace of newforms of level N is defined to be the orthog-*

onal complement of the space $S_k^{old}(N)$ in $S_k(N)$ with respect to the Petersson inner product. We shall denote the subspace of newforms of level N by $S_k^{new}(N)$. Thus we have

$$S_k(N) \;=\; S_k^{old}(N) \bigoplus S_k^{new}(N).$$

We first note the following.

Theorem 2.1.10. *The spaces $S_k^{old}(N)$ and $S_k^{new}(N)$ are stable under the action of Hecke operators $T(m)$ for all $(m, N) = 1$. Further, both the spaces $S_k^{old}(N)$ and $S_k^{new}(N)$ have orthogonal basis consisting of Hecke eigenforms.*

The following theorem captures the essential properties of Hecke eigenforms lying in the newform space.

Theorem 2.1.11. *Let $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k^{new}(N)$ be a Hecke eigenform. Then we have the following:*

1. *$a_f(1) \neq 0$.*
2. *If $a_f(1) = 1$, then $T(m)f = a_f(m)f$ for all $(m, N) = 1$.*
3. *Further assume that $M \mid N$ and $g(z) = \sum_{m=1}^{\infty} a_g(n)q^n \in S_k(M)$ is an eigenform for all Hecke operators $T(p)$ with $(p, N) = 1$. Now if the eigenvalues of g are $a_f(p)$ for all but finitely many primes p, then either $M < N$ and $g = 0$ or $M = N$ and $g = \lambda f$ for some $\lambda \in \mathbb{C}$.*

As an immediate corollary, we have the following.

Corollary 2.1.12. *The space*

$$S_k(N) \;=\; \bigoplus_{M|N} \bigoplus_{d|(N/M)} B_d(S_k^{new}(M)).$$

Further, let H be an operator on the space $S_k(N)$ which commutes with $T(m)$ for all $(m, N) = 1$. Then any Hecke eigenform is also an eigenfunction of H.

2.1.4 Hecke eigenvalues and L-functions

Let

$$f(z) \ := \ \sum_{n\geq 1} a_f(n)q^n \in S_k^{new}(N)$$

be a normalized Hecke eigenform. Then *Ramanujan-Petersson conjecture* predicts

$$|a_f(p)| \ \leq \ 2p^{(k-1)/2}$$

for any prime $(p, N) = 1$. This was proved by Deligne [19] in 1974 as a consequence of his proof of Weil's conjecture. Note that the above Ramanujan-Petersson bound implies that

$$|a_f(n)| \leq d(n)n^{(k-1)/2} \quad \text{for all} \quad (n, N) = 1,$$

where $d(n)$ denotes the number of positive divisors of n. Therefore we normalize these eigenvalues as follows:

$$\lambda_f(n) := \frac{a_f(n)}{n^{(k-1)/2}}.$$

Thus the *Ramanujan-Petersson* bounds tells us

$$\text{(2.1.3)} \qquad |\lambda_f(n)| \leq d(n) \quad \text{for all} \quad (n, N) = 1.$$

In this notation, Hecke relations turn out to be

$$\text{(2.1.4)} \qquad \lambda_f(1) = 1 \quad \text{and} \quad \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n),\\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

Using (2.1.3), for any prime $(p, N) = 1$ one can write

$$\text{(2.1.5)} \qquad \lambda_f(p) \ := \ 2\cos\theta_f(p),$$

where $\theta_f(p) \in [0, \pi]$.

Thus for any $m \in \mathbb{N}$, by the identity (2.1.4), we have

$$\lambda_f(p^m) = \begin{cases} m+1 & \text{if } \ \theta_f(p) = 0, \\ (-1)^m(m+1) & \text{if } \ \theta_f(p) = \pi, \\ \frac{\sin[(m+1)\theta_f(p)]}{\sin\theta_f(p)} & \text{if } \ \theta_f(p) \in (0,\pi). \end{cases} \tag{2.1.6}$$

Note that $\lambda_f(p^n) = 0$ for some $n \in \mathbb{N}$ if and only if $\theta_f(p)/\pi \in \mathbb{Q} \setminus \{0,1\}$. In fact, one knows the following.

Lemma 2.1.13. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform. Then for all but finitely many primes p, one has either $\theta_f(p) \in \{0, \pi/2, \pi\}$ or $\theta_f(p)/\pi \notin \mathbb{Q}$. Further, if $k \geq 4$ then we have either $\theta_f(p) \in \{0, \pi/2, \pi\}$ or $\theta_f(p)/\pi \notin \mathbb{Q}$ for all primes p with $(p, N) = 1$.*

Proof. The first part of the Lemma was proved in [68, Lemma 2.5] (also in [43, Lemma 2.2]) whereas the second part follows from [68, Lemma 2.4]. □

Note that the above lemma does not tell us how often $\theta_f(p) = \pi/2$, that is, $\lambda_f(p) = 0$. This was studied extensively by Serre [91]. To state his result, we need the following notions of CM and non-CM forms in the sense of Ribet [82].

Definition 2.1.14. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform.*

- *We say that f has complex multiplication (or, of CM type) if there exists a non-trivial Dirichlet character χ modulo D such that*

$$a_f(p)\chi(p) = a_f(p)$$

  for all but finitely many primes p.

- *A form is said to be a non-CM form or of non-CM type if it is not of CM type.*

One can show that if the level N is square-free, then there are no CM forms (see [82], Section 3 and [83], Theorem 3.9 for details). With this notion of CM forms, Serre [91] proved the following theorem.

Theorem 2.1.15. *Let $f(z) := \sum_{n\geq 1} a_f(n)q^n \in S_k^{new}(N)$ be a normalized Hecke eigenform which is a non-CM form. Then for any $0 < \delta < 1/2$, we have*

$$\#\{p \leq x \mid (p,N)=1,\ \lambda_f(p)=0\} \ \ll\ \frac{x}{(\log x)^{1+\delta}}.$$

In fact, one knows the following stronger statement.

Theorem 2.1.16. *Let $f(z) := \sum_{n\geq 1} a_f(n)q^n \in S_k^{new}(N)$ be a normalized Hecke eigenform which is a non-CM form. For integer $n \geq 1$, let*

$$P_{f,n} \ :=\ \left\{p \in \mathcal{P} \;\middle|\; p \nmid N \text{ and } \lambda_f(p^n)=0\right\}.$$

Then for any $x \geq 2$ and $0 < \delta < 1/2$, we have

$$\#(P_{f,n} \cap [1,x]) \ \ll_{f,\delta}\ \frac{x}{(\log x)^{1+\delta}}.$$

Further, if $P_f := \cup_{n\in\mathbb{N}} P_{f,n}$, then for any $x \geq 2$ and $0 < \delta < 1/2$, we have

$$\#(P_f \cap [1,x]) \ \ll_{f,\delta}\ \frac{x}{(\log x)^{1+\delta}}.$$

Here the implied constants depend on f and δ.

Proof of Theorem 2.1.16. For a proof of this theorem, we refer to Lemma 2.3 of Kowalski, Robert and Wu [43] (see also R. Murty and K. Murty [68, Lemma 2.5]). □

Now we shall introduce Rankin-Selberg L-function associated to normalized Hecke eigenforms $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ and list some of its analytic properties which will be required to prove the theorems in the later chapters. Let

(2.1.7)
$$f(z) := \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k_1}^{new}(N_1) \quad \text{and} \quad g(z) := \sum_{n=1}^{\infty} a_g(n)q^n \in S_{k_2}^{new}(N_2)$$

be normalized Hecke eigenforms. Then the Rankin-Selberg L-function of f and g, denoted by $R(f, g; s)$, is defined as follows

$$R(f, g; s) := \sum_{n \geq 1} \lambda_f(n)\lambda_g(n)n^{-s}.$$

It follows from (2.1.3) that $R(f, g; s)$ is absolutely convergent for $\Re(s) > 1$. Let us set $M := \gcd(N_1, N_2)$ and $N := \mathrm{lcm}[N_1, N_2]$. Also assume that M, N are square-free. By the works of Rankin [80] and Ogg [70, page 304], one knows that the function $\zeta_N(2s)R(f, g; s)$ is entire if $f \neq g$, where $\zeta_N(s)$ is defined by

$$\zeta_N(s) := \prod_{p \nmid N} \left(1 - p^{-s}\right)^{-1} \quad \text{for } \Re(s) > 1. \tag{2.1.8}$$

We also have the completed Rankin-Selberg L-function

$$R^*(f, g; s) := (2\pi)^{-2s}\Gamma(s+\frac{k_2 - k_1}{2})\Gamma(s+\frac{k_1 + k_2}{2}-1)\prod_{p|M}(1-c_p p^{-s})^{-1}\zeta_N(2s)R(f, g; s)$$

with $c_p = \pm 1$ depending on the forms f and g. By the works of Ogg (see [70, Theorem 6]) and Li (see [50, Theorem 2.2]), one has

$$R^*(f, g; s) = N^{1-2s}R^*(f, g; 1 - s). \tag{2.1.9}$$

2.2 Siegel modular forms

In this section, we shall list some definitions and properties of Siegel modular forms. For this section, we shall follow the approach of Andrianov [3]. For simplicity of exposition we shall restrict ourselves in the case of degree $g = 1, 2$. Let us start by defining Siegel modular forms.

For positive integer $g = 1, 2$, let $\mathcal{H}_g$ be the Siegel upper half-space of degree g which is defined as follows:

$$\mathcal{H}_g := \left\{ Z \in M_g(\mathbb{C}) \mid Z^t = Z,\ \Im(Z) > 0 \right\}.$$

Here the notation $Y > 0$ means that the real symmetric matrix Y is positive definite, that is, for any non-zero column vector x, we have $x^t Y x > 0$. Note that the space $\mathcal{H}_g$ is a $g(g+1)/2$-dimensional complex manifold. The Siegel modular group $\mathrm{Sp}_g(\mathbb{Z})$ is defined by

$$\mathrm{Sp}_g(\mathbb{Z}) := \left\{ M \in M_{2g}(\mathbb{Z}) \mid M^t J_g M = J_g \right\},$$

where $J_g := \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$ and 1_g is the $g \times g$ identity matrix. From now on, we shall denote the Siegel modular group $\mathrm{Sp}_g(\mathbb{Z})$ by Γ_g.

Remark 2.2.1. *Note that when $g = 1$, the Siegel upper half-space $\mathcal{H}_1$ denotes the complex upper half-plane $\mathcal{H}$. Also note that*

$$\begin{aligned} \Gamma_1 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \;\middle|\; ad - bc = 1 \right\}. \end{aligned}$$

Thus Γ_1 is the full modular group $\Gamma_0(1)$.

Now we are ready to define Siegel modular form.

Definition 2.2.2. *A holomorphic function $F : \mathcal{H}_g \to \mathbb{C}$ is said to be a Siegel modular form of weight k for the group Γ_g if it satisfies the following properties:*

1. *for any* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$ *and* $Z \in \mathcal{H}_g$, *we have*

$$F((AZ+B)(CZ+D)^{-1}) \;=\; det\;(CZ+D)^k F(Z).$$

2. *for any* $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$, *the function* $det(CZ+D)^{-k}f((AZ+B)(CZ+D)^{-1})$ *is bounded in any domain of the form* $\{Z \in \mathcal{H}_g \mid \Im(Z) - c1_g > 0\}$ *for any* $c > 0$.

Remark 2.2.3. *It is clear from the definition that the Siegel modular forms of degree one are nothing but elliptic modular forms which were discussed in the previous section. Thus the notion of Siegel modular form generalizes the notion of elliptic modular form.*

As in the case of elliptic modular forms, one can show that the set of all Siegel modular forms of weight k for the group Γ_g is a finite dimensional complex vector space. We shall denote this vector space by $M_k(\Gamma_g)$.

Example 2.2.4. *Let* A *be a symmetric positive definite integral matrix with even diagonal entries and det* $A = 1$. *Also assume that the order* m *of* A *is divisible by* 8. *Then for every* $g = 1, 2, ..., m$, *the theta series of* A *is defined as follows:*

$$\Theta_A^{(g)}(Z) \;:=\; \sum_{X \in M_{m,g}(\mathbb{Z})} exp(\pi i tr(X^t AXZ)) \;=\; \sum_B r_A(B) exp(\pi i tr(BZ)).$$

Here $Z \in \mathcal{H}_g$ *and* B *ranges over all integral* $g \times g$ *matrices with even diagonal entries satisfying* $B^t = B,\ B \geq 0$ *and* $r_A(B)$ *is defined as follows:*

$$r_A(B) \;:=\; \#\left\{G \in M_{m,g}(\mathbb{Z}) \mid \frac{1}{2}G^t AG = B\right\}.$$

It can be shown that this is a Siegel modular form of degree g *and weight* $m/2$ *for* Γ_g.

Like elliptic modular forms, any $F \in M_k(\Gamma_g)$ has Fourier expansion as follows:

$$F(z) \;=\; \sum_{M \geq 0} a_F(M) e^{2\pi i \mathrm{tr}(Mz)},$$

where the summation runs over the set of all symmetric $g \times g$ half-integral matrices M with integral diagonal entries satisfying $M \geq 0$, that is, the summation runs over the set

$$\left\{ M \in M_g(\mathbb{Q}) \mid M^t = M,\ m_{ii}, 2m_{ij} \in \mathbb{Z},\ M \geq 0 \right\}.$$

Similarly, we define the space $S_k(\Gamma_g)$ of Siegel cusp forms of weight k and of degree g as follows:

$$S_k(\Gamma_g) \;:=\; \{F \in M_k(\Gamma_g) \mid a_F(M) = 0 \text{ unless } M > 0\}.$$

As in the previous section, we shall investigate a basis of the space $S_k(\Gamma_g)$ which is good for our purpose.

2.2.1 Petersson inner product

On the space of Siegel modular forms, we shall define an invariant inner product such that the space of cusp forms $S_k(\Gamma_g)$ will be a Hilbert space. For $F, G \in S_k(\Gamma_g)$, consider the following differential form on $\mathcal{H}_g$:

$$\omega_k(F,G)(Z) \;:=\; F(Z)\overline{G(Z)}\,\det(\Im(Z))^k d^*Z,$$

where d^*Z is the invariant measure on $\mathcal{H}_g$ and is defined by

$$d^*Z \;:=\; \det(\Im(Z))^{-(g+1)} \prod_{\alpha \leq \beta} dx_{\alpha\beta} dy_{\alpha\beta}, \quad Z = (x_{\alpha\beta}) + i(y_{\alpha\beta}).$$

It can be shown that $\omega_k(F,G)(Z)$ is invariant under the action of Γ_g. Thus on the space $S_k(\Gamma_g)$, we have the following inner product

$$\langle F, G\rangle \ := \ \int_{\Gamma_g\backslash\mathcal{H}_g} \omega_k(F,G)(Z). \tag{2.2.1}$$

Note that if we restrict ourselves to $g=1$, the above inner product coincides with the inner product defined by (2.1.2) when $N=1$. Also note that as in the case of elliptic modular forms the integral (2.2.1) converges absolutely if at least one of F or G is a cusp form. The inner product defined above is known as the *Petersson inner product.*

2.2.2 Hecke operators on $M_k(\Gamma_g)$

Let us start by defining the *Hecke operators* on the space of Siegel modular forms $M_k(\Gamma_g)$.

Definition 2.2.5. *For any positive integer n and $g=1,2$, the Hecke operator $T_g(n)$ on the space $S_k(\Gamma_g)$ is defined by*

$$T_g(n)F \ := \ n^{gk-\frac{g(g+1)}{2}} \sum_{\gamma\in\Gamma_g\backslash\mathcal{O}_{g,n}} F \mid \gamma,$$

where

$$\mathcal{O}_{g,n} \ := \ \left\{\gamma\in M_{2g}(\mathbb{Z}) \mid \gamma^t J_g\gamma = nJ_g\right\}, \quad J_g \ := \ \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}$$

and

$$F\mid\gamma \ := \ det\,(CZ+D)^{-k}F((AZ+B)(CZ+D)^{-1}), \quad \gamma \ := \ \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here we have

Theorem 2.2.6. *Following statements are true.*

1. *The spaces $M_k(\Gamma_g)$ and $S_k(\Gamma_g)$ are stable under the action of all Hecke operators $T_g(n)$.*

2. *For $n, m \in \mathbb{N}$, we have $T_g(n)T_g(m) = T_g(m)T_g(n)$. Further, if $(n, m) = 1$, we have $T_g(n)T_g(m) = T_g(mn)$.*

3. *For any $F, G \in S_k(\Gamma_g)$ and any $m \in \mathbb{N}$, we have $\langle T_g(m)F, G\rangle = \langle F, T_g(m)G\rangle$.*

Using Theorem 2.1.4, as an immediate corollary, we have the following.

Corollary 2.2.7. *The space $S_k(\Gamma_g)$ has a basis consisting of eigenfunctions of Hecke operators $T_g(m)$ for all $m \geq 1$.*

Definition 2.2.8. *A cusp form $F \in S_k(\Gamma_g)$ is said to be a Hecke eigenform if it is an eigenfunction of all Hecke operators $T_g(m),\ m \geq 1$.*

One can define Siegel modular forms of higher levels in a similar way. As commented in [74, page 2], there is no good theory of oldforms and newforms in this context. Hence we restrict ourselves to level one in this thesis.

2.2.3 Hecke eigenvalues and L-functions

In this subsection, we shall restrict ourselves to the case $g = 2$. Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with eigenvalues $\{\mu_F(m)\}_{m\in\mathbb{N}}$, that is, $T_2(m)F = \mu_F(m)F$ for all $m \in \mathbb{N}$. Now by Theorem 2.2.6, we know that μ_F is a multiplicative function. In the space $S_k(\Gamma_2)$, there is a canonically defined subspace S_k^* which is known as *Maass subspace*. One knows that the subspace S_k^* and its orthogonal complement are stable under the action of all Hecke operators. Here we restrict ourselves in the orthogonal complement of the Maass subspace and we shall discuss the properties of Hecke eigenvalues of an eigenform lying in the *Maass subspace* in the next subsection.

If $F \in S_k(\Gamma_2)$ is a Hecke eigenform which does not lie in the *Maass subspace*, by an important work of Weissauer [99], one knows that F satisfies the *generalized Ramanujan-Petersson conjecture*, that is, for any $\epsilon > 0$, one has

$$\mu_F(n) \ll_\epsilon n^{k-3/2+\epsilon}.$$

We shall normalize these eigenvalues as follows:

$$\lambda_F(n) := \frac{\mu_F(n)}{n^{k-3/2}} \quad \text{for any} \quad n \in \mathbb{N}.$$

The normalized Hecke eigenvalues satisfies the following relation: for any prime p and any integer $n \geq 3$, we have

$$\lambda_F(p^n) = \lambda_F(p)\lambda_F(p^{n-1}) - \left[\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p}\right]\lambda_F(p^{n-2}) + \lambda_F(p)\lambda_F(p^{n-3}) \\ -\lambda_F(p^{n-4})$$

with the assumption that $\lambda_F(p^{n-m}) = 0$ for $n < m$. Following Andrianov [3], to each Hecke eigenform $F \in S_k(\Gamma_2)$, we shall attach an L-function which is now known as *Andrianov L-function or Spinor zeta function* as follows:

$$(2.2.2) \qquad Z_F(s) := \zeta(2s+1)\sum_{n=1}^{\infty}\frac{\mu_F(n)}{n^{s+k-3/2}} = \zeta(2s+1)\sum_{n=1}^{\infty}\frac{\lambda_F(n)}{n^s}.$$

The analytic properties of this function were studied by Andrianov [3] and Oda [71]. More precisely by their works we have the following theorem.

Theorem 2.2.9. *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with normalized Hecke eigenvalues $\{\lambda_F(m)\}_{m\in\mathbb{N}}$. Define $Z_F(s)$ as in (2.2.2). Then the completed L-function*

$$Z_F^*(s) := (2\pi)^{-2s}\Gamma(s+1/2)\Gamma(s+k-3/2)Z_F(s)$$

can be continued as a meromorphic function in $\mathbb{C}$ *having at most finitely many poles. Also it satisfies the functional equation* $Z_F^*(1-s) = (-1)^k Z_F^*(s)$. *Further, if* F *does not lie in the* Maass subspace *then the function* $Z_F^*(s)$ *is entire; otherwise,* $Z_F^*(s)$ *has simple poles only at* $s = 3/2, -1/2$.

Next we consider the *Rankin-Selberg L-function* associated to Siegel Hecke eigenforms. Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms with the normalized Hecke eigenvalues $\{\lambda_F(n)\}_{n\in\mathbb{N}}$ and $\{\lambda_G(n)\}_{n\in\mathbb{N}}$ respectively. Also assume that both F and G do not lie in the Maass subspace. Further, let $Z_F(s)$ and $Z_G(s)$ be the Spinor zeta functions associated to F and G respectively. We write

$$\begin{aligned}(2.2.3)\qquad\qquad & Z_F(s) \;:=\; \prod_{p\in\mathcal{P}}\prod_{i=1}^{4}\left(1-\alpha_{p,i}p^{-s}\right)^{-1} \\ \text{and}\quad & Z_G(s) \;:=\; \prod_{p\in\mathcal{P}}\prod_{i=1}^{4}\left(1-\beta_{p,i}p^{-s}\right)^{-1}.\end{aligned}$$

By the work of Weissauer [99], one knows that $|\alpha_{p,i}| = 1 = |\beta_{p,j}|$ for $1 \le i, j \le 4$. The *Rankin-Selberg L-function* $L(F\times G, s)$ associated to F and G is now defined as follows:

$$(2.2.4)\qquad L(F\times G, s) \;:=\; \prod_{p\in\mathcal{P}}\prod_{1\le i,j\le 4}\left(1-\alpha_{p,i}\beta_{p,j}p^{-s}\right)^{-1}.$$

This Euler product is absolutely convergent for $\Re(s) > 1$. In fact, by the work of Pitale, Saha and Schmidt [74, Theorem C, p. 14], we have the following theorem.

Theorem 2.2.10. *Let* $F \in S_{k_1}(\Gamma_2), G \in S_{k_2}(\Gamma_2), Z_F(s), Z_G(s)$ *and* $L(F\times G, s)$ *be as above. Then the infinite product in* (2.2.4) *converges absolutely in the region* $\Re(s) > 1$, *the function* $L(F\times G, s)$ *can be continued analytically as a meromorphic function to* $\mathbb{C}$ *and does not vanish on the line* $\Re(s) = 1$. *Moreover, the function* $L(F\times G, s)$ *is entire except in the case when* $k_1 = k_2$ *and* $\mu_F(n) = \mu_G(n)$ *for all* $n \in \mathbb{N}$. *In the last case, the function* $L(F\times G, s)$ *has a simple pole at* $s = 1$.

One can define a *naive Rankin-Selberg L-function* as follows:

$$L(F, G; s) := \sum_{n=1}^{\infty} \frac{\lambda_F(n)\lambda_G(n)}{n^s}. \tag{2.2.5}$$

Note that this series $L(F, G; s)$ is also absolutely convergent for $\Re(s) > 1$. In fact, Das, Kohnen and Sengupta [18] proved the following.

Theorem 2.2.11. *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with normalized Hecke eigenvalues $\{\lambda_F(n)\}_{n\in\mathbb{N}}$. Further assume that F does not lie in the Maass subspace. Then the function $L(F, F; s)$ can be continued as a meromorphic function to $\Re(s) > 1/2$ with only a simple pole at $s = 1$. Moreover, for sufficiently large x and any $\epsilon > 0$, we have*

$$\sum_{n\leq x} \lambda_F^2(n) = c_F x + O\left(k^{5/16} x^{31/32+\epsilon}\right),$$

where $c_F > 0$ is the residue of the L-function $L(F, F; s)$ at $s = 1$.

This suggests that the Hecke eigenvalues do not vanish often. In fact, as in the case elliptic modular forms, by a work of Kowalski and Saha [85, Appendix], we have the following.

Theorem 2.2.12. *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with eigenvalues $\mu_F(n)$ for $n \in \mathbb{N}$. Also assume that F lies in the orthogonal complement of Maass subspace. Then there exists $\delta > 0$ such that*

$$\#\{p \leq x \mid \mu_F(p) = 0\} \ll \frac{x}{(\log x)^{1+\delta}}.$$

2.2.4 Maass subspace

In this subsection, we shall consider the *Maass subspace* and the Hecke eigenvalues of an eigenform lying in this subspace.

For $g = 1$ and a normalized Hecke eigenform $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k(\Gamma_1)$, by a theorem of Deligne [19], we know that $|a_f(p)| \leq 2p^{(k-1)/2}$ for any prime p. For $g = 2$, the *generalized Ramanujan-Petersson conjecture* predicts that

$$\mu_F(p) \ll p^{k-3/2}.$$

But in 1978, Kurokawa [46] and Saito independently found examples of Siegel Hecke eigenforms which contradicts this expectation. Kurokawa explicitly worked out nine counterexamples to the *generalized Ramanujan-Petersson conjecture*. Based on his computation, Kurokawa [46] suggested the following conjecture.

Conjecture 2.2.13. *Let $k \geq 10$ be an even integer. Then there exists a one-one $\mathbb{C}$-linear map $\psi_k : S_{2k-2}(\Gamma_1) \to S_k(\Gamma_2)$ with the following properties: if $f \in S_{2k-2}(\Gamma_1)$ is a normalized Hecke eigenform with eigenvalues $\{a_f(n)\}_{n\in\mathbb{N}}$, then $F := \psi_k(f)$ is a Hecke eigenform (with the eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$) which satisfies the following:*

$$\zeta(2s-2k+4)\sum_{n=1}^{\infty}\frac{\mu_F(n)}{n^s} = \zeta(s-k+2)\zeta(s-k+1)\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s}. \tag{2.2.6}$$

Soon after this work, Maass [54, 55, 56] studied a subspace S_k^* of the space $S_k(\Gamma_2)$ which is defined as follows:

$$S_k^* := \left\{ F \in S_k(\Gamma_2) \mid a_F\begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} = \sum_{d|(n,m,r)} d^{k-1} a_F \begin{pmatrix} nm/d^2 & r/2d \\ r/2d & 1 \end{pmatrix} \right. \\ \left. \forall \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} > 0 \right\},$$

where $a_F(M)$ is the Fourier coefficient of $F \in S_k(\Gamma_2)$. In his works, the space S_k^* was called as *Spezialschar* but now it is known as *Maass subspace.* He also established a relation between the Maass subspace S_k^* and the image of the conjectural map $\psi_k : S_{2k-2} \to S_k(\Gamma_2)$. More precisely, by the works of Maass [54, 55, 56], Andrianov

[4] and Zagier [103] (also see [20]), we have the following theorem.

Theorem 2.2.14. *The Maass subspace S_k^* is stable under the action of Hecke operators $T_2(m)$ for all $m \in \mathbb{N}$ and is spanned by Hecke eigenforms. These are in one-to-one correspondence with normalized Hecke eigenforms $f \in S_{2k-2}(\Gamma_1)$, the correspondence being such that* (2.2.6) *holds.*

2.3 A Comparison

Here we give a table indicating similarities and differences between the theories of elliptic modular forms and of Siegel modular forms of degree two.

	Elliptic modular forms	Siegel modular forms of degree two
1.	The full modular group $\mathrm{SL}_2(\mathbb{Z})$ is defined as $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \mid ad - bc = 1\right\}$.	The full Siegel modular group $\mathrm{Sp}_2(\mathbb{Z})$ is defined as $\left\{M := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4(\mathbb{Z}) \mid M^t J_2 M = J_2\right\}$, where $A, B, C, D \in M_2(\mathbb{Z})$.
2.	The upper half plane $\mathcal{H}$ is defined as follows: $\mathcal{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.	The Siegel upper half space of degree two is defined as follows: $\mathcal{H}_2 := \{Z \in M_2(\mathbb{C}) \mid Z^t = Z,\ \Im(Z) > 0\}$. Here $Y > 0$ means that it is positive definite.
3.	The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ is defined as follows: $\mathrm{SL}_2(\mathbb{Z}) \times \mathcal{H} \to \mathcal{H}$ $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto \frac{az+b}{cz+d}$.	The group $\mathrm{Sp}_2(\mathbb{Z})$ acts on $\mathcal{H}_2$ in a similar way: $\mathrm{Sp}_2(\mathbb{Z}) \times \mathcal{H}_2 \to \mathcal{H}_2$ $\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}, Z\right) \mapsto (AZ+B)(CZ+D)^{-1}$.

	Elliptic modular forms	Siegel modular forms of degree two
4.	The set of modular forms $M_k(\mathrm{SL}_2(\mathbb{Z}))$ of weight k for $\mathrm{SL}_2(\mathbb{Z})$ is a finite dimensional complex vector space.	The space of Siegel modular forms $M_k(\mathrm{Sp}_2(\mathbb{Z}))$ of weight k for $\mathrm{Sp}_2(\mathbb{Z})$ is a finite dimensional complex vector space.
5.	Let $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Then it has Fourier expansion as follows: $f(z) := \sum_{n=0}^{\infty} a_f(n)e^{2\pi inz}$.	Any $F \in M_k(\mathrm{Sp}_2(\mathbb{Z}))$ has Fourier expansion as follows: $F(Z) := \sum_{N\geq 0} a_F(N)e^{2\pi i \mathrm{tr}(NZ)}$. Here the above summation runs over the set of all 2×2 symmetric positive semi-definite half-integral matrices having integral diagonal entries.
6.	The space of cups forms $S_k(\mathrm{SL}_2(\mathbb{Z}))$ can be defined as follows: $\{f \in M_k(\mathrm{SL}_2(\mathbb{Z})) \mid a_f(0) = 0\}$.	One can define the space of cups forms $S_k(\mathrm{Sp}_2(\mathbb{Z}))$ by $\{F \in M_k(\mathrm{Sp}_2(\mathbb{Z})) \mid a_F(N) = 0$ if N is not positive definite$\}$.
7.	One has well defined Hecke operators on the space $M_k(\mathrm{SL}_2(\mathbb{Z}))$. Both the spaces $M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $S_k(\mathrm{SL}_2(\mathbb{Z}))$ are stable under the action of Hecke operators.	One has the notion of Hecke operators on the space $M_k(\mathrm{Sp}_2(\mathbb{Z}))$. As in the case of elliptic modular forms, both the spaces $M_k(\mathrm{Sp}_2(\mathbb{Z}))$ and $S_k(\mathrm{Sp}_2(\mathbb{Z}))$ are stable under the action of Hecke operators.
8.	The space $S_k(\mathrm{SL}_2(\mathbb{Z}))$ is a finite dimensional Hilbert space with respect to the Petersson inner product. Also the Hecke operators are Hermitian and commute with each other. Hence	One can define an inner product known as Petersson inner product on the space $S_k(\mathrm{Sp}_2(\mathbb{Z}))$. Also in this case the Hecke operators are Hermitian and commute with each other. Hence there exists a

	Elliptic modular forms	Siegel modular forms of degree two
	there exists a basis of $S_k(\mathrm{SL}_2(\mathbb{Z}))$ consisting of eigenvectors of all Hecke operators.	basis of $S_k(\mathrm{Sp}_2(\mathbb{Z}))$ consisting of eigenvectors of all Hecke operators.
9.	For higher level elliptic modular forms, the theory of newforms is well developed. Further, strong multiplicity one theorem holds for Hecke eigenforms which are newforms.	For higher level Siegel modular forms, there is also a theory of newforms. For the status of multiplicity one theorem see Schmidt [89] and Atobe [7].
10.	For a normalized Hecke eigenform Fourier coefficients and Hecke eigenvalues are equal.	There are no good relation between Fourier coefficients and Hecke eigenvalues. The Fourier coefficients are indexed by certain matrices, hence we do not know what would be the analogue of the notion of multiplicative function here.
11.	Hecke eigenvalues satisfy Ramanujan-Petersson bound. Further, Hecke eigenvalues are multiplicative and satisfy a recurrence relation of degree two.	These Hecke eigenvalues also satisfy the generalised Ramanujan-Petersson bound if the corresponding eigenforms do not lie in the Maass subspace. In this case, Hecke eigenvalues are multiplicative but they satisfy a recurrence relation of degree four.
12.	With each normalized Hecke eigenform, one can attach an L-function. It is known that the completed L-function satisfies functional equation	In this context, one can attach an L-function with a Hecke eigenform. This L-function is known as spinor zeta function or Andrianov L-function. One

	Elliptic modular forms	Siegel modular forms of degree two
	and can be extended analytically to $\mathbb{C}$ as an entire function.	also knows that the completed spinor zeta function satisfies functional equation and can be extended analytically to $\mathbb{C}$ as a meromorphic function with at most finitely many poles. Further, it is entire if and only if Hecke eigenform does not lie in the Maass subspace.
13.	Analytic properties of Rankin-Selberg L-function attached to two normalized Hecke eigenforms f and g are known for arbitrary level.	Analytic properties of Rankin-Selberg L-function are known only for level one.
14.	It is known that the Hecke eigenvalues change sign infinitely often and satisfy Sato-Tate distribution.	The Hecke eigenvalues of a Hecke eigenform lying in the Maass subspace are always positive. Further, if a Hecke eigenform does not lie in the Maass subspace, then its Hecke eigenvalues change sign infinitely often. But we do not know analogous Sato-Tate distribution in this context.

2.4 $\mathfrak{B}$-free numbers

In order to estimate the gap between two consecutive square-free integers, Erdös [22] introduced the notion of $\mathfrak{B}$-free numbers as follows:

Definition 2.4.1. *Let* $\mathfrak{B} := \{b_k \in \mathbb{N} \mid 1 < b_1 < b_2 < \cdots\}$ *be an infinite set such* *that*

$$(b_i, b_j) = 1 \quad \textit{for} \ \ i \neq j \quad \ \textit{and} \quad \sum_{i \geq 1} \frac{1}{b_i} \ < \ \infty.$$

We say that a number $n \in \mathbb{N}$ is $\mathfrak{B}$-free if it is not divisible by any element of the set $\mathfrak{B}$.

In the same paper, he proved the existence of an absolute constant $\theta < 1$ such that for sufficiently large $x > 0$ every intervals $[x, x + x^\theta)$ contains a $\mathfrak{B}$-free number. Further, he conjectured the following:

Conjecture 2.4.2. *For any $\theta > 0$, there exists $N_{\mathfrak{B}}$ such that for any $x > N_{\mathfrak{B}}$ the interval $[x, x + x^\theta)$ contains at least one $\mathfrak{B}$-free number.*

A lot of work has been done to improve the values of θ. For instance [$\theta > 1/2$, [97]], [$\theta > 9/20$, [9]], [$\theta > 17/41$, [100]], [$\theta > 33/80$, [104]] and [$\theta > 40/97$, [87]] to name a few. In this direction, Granville [25] showed that Conjecture 2.4.2 is true in the case of square-free integers if one assumes that ABC Conjecture is true. Unconditionally, Filaseta and Trifonov [23] proved that θ can be chosen to be any number $> 1/5$ in the case of square-free numbers.

One of the most interesting applications of $\mathfrak{B}$-free numbers is in the study of non-vanishing of Hecke eigenvalues of modular forms. In this connection, Kowalski, Robert and Wu [43] considered a special set of $\mathfrak{B}$-free numbers which occurs in the study of non-vanishing of Hecke eigenvalues of modular forms. We discuss it briefly here.

Let $\mathfrak{P}$ be a subset of $\mathcal{P}$ such that

$$\#\left(\mathfrak{P} \cap [1, x]\right) \ \ll_{\mathfrak{P}} \ \frac{x^\rho}{(\log x)^\eta} \tag{2.4.1}$$

where $\rho \in [0, 1]$ and η is non-negative real numbers with the assumption that $\eta > 1$

when $\rho = 1$. We define

$$\mathfrak{B}_{\mathfrak{P}} := \mathfrak{P} \cup \left\{p^2 \mid p \in \mathcal{P} - \mathfrak{P}\right\}. \tag{2.4.2}$$

Write $\mathfrak{B}_{\mathfrak{P}} = \{b_i \mid i \in \mathbb{N}\}$. Note that $(b_i, b_j) = 1$ for all $b_i, b_j \in \mathfrak{B}_{\mathfrak{P}}$ with $b_i \neq b_j$. To show $\sum_{i \in \mathbb{N}} 1/b_i < \infty$, it is enough to show that $\sum_{p \in \mathfrak{P}} 1/p < \infty$. Applying equation (2.4.1) and partial summation formula, we have

$$\sum_{\substack{p \le x, \\ p \in \mathfrak{P}}} \frac{1}{p} = \frac{1}{x} \sum_{\substack{p \le x, \\ p \in \mathfrak{P}}} 1 + \int_2^x \frac{1}{t^2} (\sum_{\substack{p \le t, \\ p \in \mathfrak{P}}} 1) dt \ll_{\mathfrak{P}} \frac{x^{\rho - 1}}{(\log x)^{\eta_\rho}} + \int_2^x \frac{t^{\rho-2}}{(\log t)^{\eta_\rho}} dt \ll_{\mathfrak{P}} 1.$$

With these notations, Kowalski, Robert and Wu (see Corollary 10 of [43]) proved the following theorem.

Theorem 2.4.3. *Let $\mathfrak{P}$ and $\mathfrak{B}_{\mathfrak{P}}$ be as above. Then for any $\epsilon > 0$ there exists $x_0(\mathfrak{P}, \epsilon) > 0$ such that*

$$\# \{x < n \le x + y \mid n \textit{ is } \mathfrak{B}_{\mathfrak{P}}\textit{-free}\} \gg_{\mathfrak{P}, \epsilon} y$$

for any $x > x_0(\mathfrak{P}, \epsilon)$ and $y \ge x^{\theta(\rho) + \epsilon}$, where

$$\theta(\rho) := \begin{cases} \frac{1}{4} & \textit{if } 0 \le \rho \le \frac{1}{3}; \\ \frac{10\rho}{19\rho + 7} & \textit{if } \frac{1}{3} < \rho \le \frac{9}{17}; \\ \frac{3\rho}{4\rho + 3} & \textit{if } \frac{9}{17} < \rho \le \frac{15}{28}; \\ \frac{5}{16} & \textit{if } \frac{15}{28} < \rho \le \frac{5}{8}; \\ \frac{22\rho}{24\rho + 29} & \textit{if } \frac{5}{8} < \rho \le \frac{9}{10}; \\ \frac{7\rho}{9\rho + 8} & \textit{if } \frac{9}{10} < \rho \le 1. \end{cases} \tag{2.4.3}$$

In 2005, Alkan and Zaharescu [1] considered $\mathfrak{B}$-free numbers in arithmetic progressions and proved the existence of $\mathfrak{B}$-free numbers in short arithmetic progressions. In order to study non-vanishing of Hecke eigenvalues of modular forms, Wu and Zhai (see Proposition 4.1 of [102]) have considered the set $\mathfrak{B}_{\mathfrak{P}}$ and studied the properties of $\mathfrak{B}_{\mathfrak{P}}$-free numbers in short arithmetic progression. In particular, they proved the following theorem.

Theorem 2.4.4. *Let $\mathfrak{B}_{\mathfrak{P}}$ be as in* (2.4.2)*. Then for any $\epsilon > 0$ there exists $x_0(\mathfrak{P}, \epsilon) > 0$ such that for any $x \geq x_0(\mathfrak{P}, \epsilon)$, $y \geq x^{\psi(\rho)+\epsilon}$ and $1 \leq a \leq q \leq x^{\epsilon}$ with $(a, q) = 1$, we have*

$$\#\left\{x < n \leq x + y \mid n \textit{ is } \mathfrak{B}_{\mathfrak{P}}\textit{-free and } n \equiv a \ (\textit{mod } q)\right\} \gg_{\mathfrak{P},\epsilon} \frac{y}{q},$$

where

$$\psi(\rho) := \begin{cases} \frac{29\rho}{46\rho+19} & \textit{if } \frac{190}{323} < \rho \leq \frac{166}{173}; \\ \frac{17\rho}{26\rho+12} & \textit{if } \frac{166}{173} < \rho \leq 1. \end{cases} \tag{2.4.4}$$

Chapter 3

Sign changes of Hecke eigenvalues of modular forms

Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms with Fourier expansions as follows:

$$\begin{aligned} (3.0.1) \qquad\qquad f(z) &:= \sum_{n=1}^{\infty} a_f(n) q^n = \sum_{n=1}^{\infty} n^{(k_1-1)/2} \lambda_f(n) q^n \\ \text{and} \quad g(z) &:= \sum_{n=1}^{\infty} a_g(n) q^n = \sum_{n=1}^{\infty} n^{(k_2-1)/2} \lambda_g(n) q^n. \end{aligned}$$

The question of sign changes of $\{a_f(n)\}_{n\in\mathbb{N}}$ has been studied extensively by several mathematicians (see for example [13], [38], [39], [41], [48], [57], [64]). In this chapter, we shall investigate sign changes of $\{a_f(n)a_g(n)\}_{n\in\mathbb{N}}$ when $f \neq g$.

We start by proving a variant of classical *Sturm's theorem*. The simultaneous sign changes of Hecke eigenvalues in short intervals are considered in section 3.2. In sections 3.3 and 3.4, we shall study distribution of Hecke eigenvalues of non-CM forms and CM forms respectively. In these two sections, we shall list some of the results from the literature regarding the distribution of these Hecke eigenvalues and derive certain variants of these results which are required to prove our theorems.

Moreover, we shall derive some new results in these sections. In the last section, we link the question of sign changes to multiplicity one theorem and use this in the study of sign changes of Hecke eigenvalues. This chapter is based on two joint works; first one is joint paper with Gun and Kumar [28] and second one is joint paper with Gun [30].

3.1 A variant of Sturm's theorem

As mentioned in the introduction of this thesis, we know that the signs of the Hecke eigenvalues determine normalized Hecke eigenforms uniquely. Here we address the following question.

Question: Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms with eigenvalues $\{a_f(n)\}_{n\in\mathbb{N}}$ and $\{a_g(n)\}_{n\in\mathbb{N}}$ respectively. Can we determine whether $f = g$ by the signs of their first few Hecke eigenvalues?

We answer this question positively by proving the following theorem.

Theorem 3.1.1. *Let $N_1,\ N_2$ be square-free integers, $N := lcm\,[N_1, N_2]$ and $f \in S_{k_1}^{new}(N_1),\ g \in S_{k_2}^{new}(N_2)$ be two distinct normalized Hecke eigenforms with the Fourier expansions as in* (3.0.1). *Then for any $\epsilon > 0$, there exists a prime power $p^\alpha,\ \alpha \le 2$ with*

$$p^\alpha \ll_\epsilon max\Big\{exp\Big[c\log^2\Big(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)}\Big)\Big], \left[N^2\left(1 + |k_2 - k_1|\right)(k_1 + k_2)\right]^{1+\epsilon}\Big\}$$

such that $a_f(p^\alpha)a_g(p^\alpha) < 0$. Here $c > 0$ is an absolute constant and $\mathfrak{q}(f), \mathfrak{q}(g)$ are analytic conductors of the Rankin-Selberg L-functions of f and g respectively.

To prove this theorem, we rely on an idea of Iwaniec, Kohnen and Sengupta [36], analytic properties of Rankin-Selberg L-functions associated to f and g and the prime number theorem for these L-functions. The above theorem can be compared with the results of Lau, Liu and Wu [48], Kohnen [37], Kowalski, Michel and Vanderkam [42], R. Murty [65] and Sengupta [90].

3.1.1 Proof of Theorem 3.1.1

Through out we assume N_1, N_2, N are as in Theorem 3.1.1, $M := (N_1, N_2)$ and $1 < k_1 \leq k_2$ are positive integers. We start by proving the following propositions.

Proposition 3.1.2. *Let N_1, N_2, N, M, k_1 and k_2 be as before. Also let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms. Then for any $t \in \mathbb{R}$ and $\epsilon > 0$, one has*

$$\begin{aligned}
&\zeta_N(2+2\epsilon+2it)R(f,g;1+\epsilon+it) \ll_\epsilon 1,\\
&\zeta_N(-2\epsilon+2it)R(f,g;-\epsilon+it) \ll_\epsilon \left[N^2\left(1+\frac{k_2-k_1}{2}\right)\left(\frac{k_1+k_2}{2}\right)|1+it|^2\right]^{(1+2\epsilon)}
\end{aligned}$$

where $\zeta_N(s)$ is defined in (2.1.8).

Proof of Proposition 3.1.2. Since

$$|\zeta_N(2+2\epsilon+2it)| \leq \zeta(2+2\epsilon)\cdot\prod_p(1+p^{-2-2\epsilon}) = \frac{\zeta^2(2+2\epsilon)}{\zeta(4+4\epsilon)} \ll 1$$

$$\text{and} \quad |R(f,g;1+\epsilon+it)| \leq \sum_{n\geq 1}\left|\frac{\lambda_f(n)\lambda_g(n)}{n^{1+\epsilon+it}}\right| \leq \sum_{n\geq 1}\frac{d(n)^2}{n^{1+\epsilon}} \ll_\epsilon 1,$$

we have the first inequality. To derive the second inequality, we use functional equation. From the functional equation (2.1.9), we have

$$\begin{aligned}
(3.1.1)\quad \zeta_N(2-2s)\cdot R(f,g;1-s) &= \left(\frac{N}{4\pi^2}\right)^{2s-1}\cdot\frac{\Gamma(s+\frac{k_2-k_1}{2})}{\Gamma(1-s+\frac{k_2-k_1}{2})}\\
&\cdot\frac{\Gamma(s+\frac{k_1+k_2}{2}-1)}{\Gamma(-s+\frac{k_1+k_2}{2})}\cdot\prod_{p|M}\left(\frac{1-c_pp^{s-1}}{1-c_pp^{-s}}\right)\cdot\zeta_N(2s)\cdot R(f,g;s).
\end{aligned}$$

Using Stirling's formula (see page 57 of [34]), for any $t \in \mathbb{R}$, we have

$$\left|\frac{\Gamma(1+\frac{k_2-k_1}{2}+\epsilon+it)}{\Gamma(\frac{k_2-k_1}{2}-\epsilon+it)}\right| \ll_\epsilon \left(1+\frac{k_2-k_1}{2}\right)^{1+2\epsilon} |1+it|^{1+2\epsilon}$$

$$\text{and} \quad \left|\frac{\Gamma(\frac{k_1+k_2}{2}+\epsilon+it)}{\Gamma(\frac{k_1+k_2}{2}-1-\epsilon+it)}\right| \ll_\epsilon \left(\frac{k_1+k_2}{2}\right)^{1+2\epsilon} |1+it|^{1+2\epsilon}.$$

For all $t \in \mathbb{R}$, we also have

$$\left|\prod_{p|M}(1-c_p p^{-1-\epsilon-it})^{-1}\right| = \left|\prod_{p|M}\sum_{m\geq 0}(c_p p^{-1-\epsilon-it})^m\right| \leq \prod_{p|M}\sum_{m\geq 0}(p^{-1-\epsilon})^m \ll_\epsilon 1$$

and

$$\left|\prod_{p|M}(1-c_p p^{\epsilon+it})\right| = \prod_{p|M}\left|1-c_p p^{\epsilon+it}\right| \leq \prod_{p|M}(1+p^\epsilon) \leq \prod_{p|M} p^{1+\epsilon} = M^{1+\epsilon}.$$

Putting $s = 1+\epsilon-it$ in equation (3.1.1) and using the above estimates along with the first inequality, we get the second inequality. $\square$

The next proposition provides convexity bound for the Rankin-Selberg L-function $R(f,g;s)$.

Proposition 3.1.3. *Let $N_1, N_2, N, M, k_1, k_2, f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be as in Proposition 3.1.2. Then for any $t \in \mathbb{R}$, $\epsilon > 0$ and $1/2 < \sigma < 1$, one has*

$$R(f,g;\sigma+it) \ll_\epsilon N^{2(1-\sigma+\epsilon)}\left(1+\frac{k_2-k_1}{2}\right)^{1-\sigma+\epsilon}\left(\frac{k_1+k_2}{2}\right)^{1-\sigma+\epsilon}\big(3+|t|\big)^{2(1-\sigma)+\epsilon}.$$

To prove this proposition, we shall use the following strong convexity principle due to Rademacher [76].

Proposition 3.1.4. *Let g be a continuous function on the closed strip $a \leq \sigma \leq b$,* *holomorphic and of finite order on $a < \sigma < b$. Further suppose that*

$$|g(a+it)| \ \leq \ E|P+a+it|^{\alpha}, \qquad |g(b+it)| \ \leq \ F|P+b+it|^{\beta}$$

where E, F are positive constants and P, α, β are real constants satisfying

$$P + a \ > \ 0, \qquad \alpha \geq \beta.$$

Then for all $a < \sigma < b$ and for all $t \in \mathbb{R}$, we have

$$|g(\sigma+it)| \ \leq \ \left(E|P+\sigma+it|^{\alpha}\right)^{\frac{b-\sigma}{b-a}} \left(F|P+\sigma+it|^{\beta}\right)^{\frac{\sigma-a}{b-a}}.$$

We are now ready to prove Proposition 3.1.3.

Proof of Proposition 3.1.3. We apply Proposition 3.1.4 with

$$\begin{aligned} a \ &= \ -\epsilon, \qquad b \ = \ P \ = \ 1+\epsilon, \qquad F \ = \ C_2, \\ E \ &= \ C_1 N^{2+4\epsilon} \left(1 + \frac{k_2 - k_1}{2}\right)^{1+2\epsilon} \left(\frac{k_1+k_2}{2}\right)^{1+2\epsilon}, \ \alpha \ = \ 2+4\epsilon, \ \beta \ = \ 0, \end{aligned}$$

where C_1, C_2 are absolute constants depending only on ϵ. Thus for any $-\epsilon < \sigma < 1+\epsilon$, we have

$$\begin{aligned} \zeta_N(2\sigma+2it) R(f,g;\sigma+it) \ \ll_{\epsilon} \ & \left[N^{\frac{2+4\epsilon}{1+2\epsilon}} \left(1 + \frac{k_2-k_1}{2}\right)\left(\frac{k_1+k_2}{2}\right)\right]^{1-\sigma+\epsilon} \\ & \cdot \left(1+\sigma+\epsilon+|t|\right)^{2(1-\sigma+\epsilon)}. \end{aligned}$$

Note that for $1/2 < \sigma < 1+\epsilon$, one knows

$$|\zeta_N(2\sigma+2it)|^{-1} \ \ll_{\epsilon} \ \log\log(N+2) \cdot |1+it|^{\epsilon}.$$

Combining all together, we get Proposition 3.1.3. $\square$

As an immediate corollary, we have the following.

Corollary 3.1.5. *Let* $N_1, N_2, N, M, k_1, k_2, f \in S_{k_1}^{new}(N_1)$ *and* $g \in S_{k_2}^{new}(N_2)$ *be as in Proposition 3.1.3. Then for any* $t \in \mathbb{R}$ *and any* $\epsilon > 0$*, one has*

$$R(f, g; 3/4 + it) \ \ll_\epsilon \ \left[N^2 \left(1 + \frac{k_2 - k_1}{2}\right) \left(\frac{k_1 + k_2}{2}\right)\right]^{1/4+\epsilon} \big(3 + |t|\big)^{1/2+\epsilon}.$$

Proposition 3.1.6. *Let* N_1, N_2, N, M, k_1 *and* k_2 *be as in Proposition 3.1.3. Also assume that* $f \in S_{k_1}^{new}(N_1)$ *and* $g \in S_{k_2}^{new}(N_2)$ *are distinct normalized Hecke eigenforms. Then for any* $\epsilon > 0$*, one has*

(3.1.2)
$$\sum_{\substack{n \le x, \\ (n,N)=1 \\ n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \log^2(x/n) \ \ll_\epsilon \ \left[N^2 \left(1 + \frac{k_2 - k_1}{2}\right) \left(\frac{k_1 + k_2}{2}\right)\right]^{1/4+\epsilon} x^{3/4}.$$

Proof of Proposition 3.1.6. Using *Deligne's bound*, we know that

$$\lambda_f(n)\lambda_g(n) \ \ll_\epsilon \ n^\epsilon$$

for any $\epsilon > 0$. Hence by Perron's summation formula (see page 56 and page 67 of [66]), we have

$$\sum_{\substack{n \le x \\ (n,N)=1 \\ n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \log^2(x/n) \ = \ \frac{1}{\pi i} \int_{1+\epsilon-i\infty}^{1+\epsilon+i\infty} R^\flat(f, g; s) \frac{x^s}{s^3} \, ds$$

where

$$R^\flat(f, g; s) \ = \ \prod_{p \nmid N} \left(1 + \frac{\lambda_f(p)\lambda_g(p)}{p^s}\right), \quad \Re(s) > 1.$$

Further, we have

$$H(s)R(f, g; s) \ = \ R^\flat(f, g; s), \tag{3.1.3}$$

where $H(s)$ is a Dirichlet series which converges normally in $\Re(s) > 1/2$. Now we shift the line of integration to $\Re(s) = 3/4$. Observing that there are no singularities in the vertical strip bounded by the lines with $\Re(s) = 1 + \epsilon$ and $\Re(s) = 3/4$ and using Proposition 3.1.3 along with the identity (3.1.3), we have

$$\begin{aligned}
\left| \frac{1}{\pi i} \int_{3/4+iT}^{1+\epsilon+iT} R^{\flat}(f,g;s) \frac{x^s}{s^3} ds \right| &\ll \frac{x^{1+\epsilon}}{T^3} \int_{3/4}^{1+\epsilon} |R^{\flat}(f,g;\sigma+iT)| d\sigma \\
&= \frac{x^{1+\epsilon}}{T^3} \int_{3/4}^{1+\epsilon} |H(\sigma+iT) R(f,g;\sigma+iT)| d\sigma \\
&\ll_{N,k_1,k_2} \frac{x^{1+\epsilon}}{T^3} \cdot T^{2(1-3/4)+\epsilon} \int_{3/4}^{1+\epsilon} d\sigma \to 0,
\end{aligned}$$

as $T \to \infty$ and similarly

$$\left| \frac{1}{\pi i} \int_{1+\epsilon-iT}^{3/4-iT} R^{\flat}(f,g;s) \frac{x^s}{s^3} ds \right| \to 0$$

as $T \to \infty$. Hence we have

$$\sum_{\substack{n \le x, \\ (n,N)=1 \\ n \text{ square-free}}} \lambda_f(n) \lambda_g(n) \log^2(x/n) = \frac{1}{\pi i} \int_{3/4-i\infty}^{3/4+i\infty} R^{\flat}(f,g;s) \frac{x^s}{s^3}\, ds.$$

The above observations combined with Corollary 3.1.5 then imply that

$$\sum_{\substack{n \le x, \\ (n,N)=1 \\ n \text{ square-free}}} \lambda_f(n) \lambda_g(n) \log^2(x/n) \ll_{\epsilon} N^{1/2+\epsilon} \left(1 + \frac{k_2 - k_1}{2}\right)^{1/4+\epsilon} \left(\frac{k_1 + k_2}{2}\right)^{1/4+\epsilon} x^{3/4}.$$

This completes the proof of the proposition. $\square$

Our next proposition will play a key role in proving Theorem 3.1.1.

Proposition 3.1.7. *For square-free integers N_1, N_2, let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms such that $f \neq g$ and $N :=$ lcm $[N_1, N_2]$. Further for any positive integer $\alpha \le 2$, let us assume that $\lambda_f(p^\alpha)\lambda_g(p^\alpha) \ge 0$ for all*

$p^\alpha \leq x$. *Then for any* $x \;\geq\; exp[c\log^2(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)})]$, *we have*

$$\sum_{\substack{n\leq x, \\ (n,N)=1 \\ n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \;\gg\; \frac{x}{\log^2 x},$$

where $c > 0$ *is an absolute constant and* $\mathfrak{q}(f)$, $\mathfrak{q}(g)$ *are analytic conductors of the* L*-functions of* $f \otimes f$ *and* $g \otimes g$ *respectively.*

Remark 3.1.8. *Using the functional equation* (2.1.9), *we see that*

$$\mathfrak{q}(f) \;\ll\; k_1^2 N_1^2 \log\log N_1 \quad \textit{and} \quad \mathfrak{q}(g) \;\ll\; k_2^2 N_2^2 \log\log N_2. \tag{3.1.4}$$

Proof of Proposition 3.1.7. Using Hecke relation (2.1.4), for any prime $(p, N) = 1$, we know that

$$\lambda_f(p^2)\lambda_g(p^2) \;=\; [\lambda_f(p)\lambda_g(p)]^2 \;-\; \lambda_f(p)^2 \;-\; \lambda_g(p)^2 \;+\; 1.$$

By hypothesis, one has $\lambda_f(p^2)\lambda_g(p^2) \geq 0$ for all $p \leq \sqrt{x}$. Hence for any $p \leq \sqrt{x}$ and $(p, N) = 1$, we have

$$\lambda_f(p)^2\lambda_g(p)^2 \;\geq\; \lambda_f(p)^2 \;+\; \lambda_g(p)^2 \;-\; 1.$$

This implies that

$$\sum_{\substack{p\leq\sqrt{x}, \\ (p,N)=1}} \lambda_f(p)^2\lambda_g(p)^2 \;\geq\; \sum_{\substack{p\leq\sqrt{x}, \\ (p,N)=1}} \lambda_f(p)^2 \;+\; \sum_{\substack{p\leq\sqrt{x}, \\ (p,N)=1}} \lambda_g(p)^2 \;-\; \sum_{\substack{p\leq\sqrt{x}, \\ (p,N)=1}} 1.$$

We would like to apply the prime number theorem to the L-functions of $f \otimes f$ and $g \otimes g$ respectively (see [35], pages 94-95, 110-111 for further details). In order to do this, we need to verify that there is no zero of $L(s, f \otimes f) \;:=\; \zeta_N(2s)R(f, f; s)$ in the region

$$\Re(s) \geq \ 1 \ - \ \frac{c_2}{\log \ [\mathfrak{q}(f)(|\Im(s)|+3)]}, \tag{3.1.5}$$

where $c_2 > 0$ is an absolute constant with at most one exceptional real zero $\beta_{f\otimes f}$ in this region. Further, we need to verify the estimate

$$\sum_{n\leq x} |\Lambda_{f\otimes f}(n)|^2 \ \ll \ x \log^2 \ (x\mathfrak{q}(f)), \tag{3.1.6}$$

where the implied constant is absolute. We see that the identity (3.1.5) holds by Theorem 5.44 of [35] (see pages 139-140 of [35] for further details) and the fact that the Riemann zeta function has the following zero-free region

$$\Re(s) \ \geq \ 1 - \frac{c'}{\log \ [|\Im(s)|+3]},$$

where $c' > 0$ is an absolute constant with no exceptional real zero. Using the *Ramanujan-Petersson* bound, we can easily deduce the identity (3.1.6). Now by applying the prime number theorem to the L-functions of $f\otimes f$ and $g\otimes g$ respectively, we see that

$$\sum_{\substack{p\leq\sqrt{x}, \\ (p,N)=1}} \lambda_f(p)^2\lambda_g(p)^2 \ \geq \ \frac{\sqrt{x}}{\log x}$$

provided $x \geq exp(c\log^2(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)}))$, where $c > 0$ is an absolute constant and $\mathfrak{q}(f), \mathfrak{q}(g)$ are as in equation (3.1.4). Using the hypothesis

$$\lambda_f(p)\lambda_g(p) \ \geq \ 0 \quad \text{and} \quad \lambda_f(p^2)\lambda_g(p^2) \ \geq \ 0$$

for all $p, p^2 \leq x$ and assuming that $x \geq exp(c\log^2(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)}))$, we have

$$\begin{aligned}\sum_{\substack{n\leq x,\\(n,N)=1\\n \text{ square-free}}} \lambda_f(n)\lambda_g(n) &\geq \frac{1}{2}\sum_{\substack{p,q\leq\sqrt{x},\\(pq,N)=1,\\p\neq q}} \lambda_f(pq)\lambda_g(pq)\\ &= \frac{1}{2}\left(\sum_{\substack{p\leq\sqrt{x},\\(p,N)=1}} \lambda_f(p)\lambda_g(p)\right)^2 - \frac{1}{2}\sum_{\substack{p\leq\sqrt{x},\\(p,N)=1}} \lambda_f(p)^2\lambda_g(p)^2.\end{aligned}$$

Now using *Deligne's bound*, we get

$$\begin{aligned}\sum_{\substack{n\leq x,\\(n,N)=1\\n \text{ square-free}}} \lambda_f(n)\lambda_g(n) &\geq \frac{1}{2}\left(\sum_{\substack{p\leq\sqrt{x},\\(p,N)=1}} \lambda_f(p)\lambda_g(p)\frac{\lambda_f(p)\lambda_g(p)}{4}\right)^2 - 8\sum_{\substack{p\leq\sqrt{x},\\(p,N)=1}} 1\\ &= \frac{1}{32}\left(\sum_{\substack{p\leq\sqrt{x},\\(p,N)=1}} \lambda_f(p)^2\lambda_g(p)^2\right)^2 + O\left(\frac{\sqrt{x}}{\log x}\right)\\ &\gg \frac{x}{\log^2 x}.\end{aligned}$$

This completes the proof of the proposition. $\square$

We are now in a position to complete the proof of Theorem 3.1.1.

Proof of Theorem 3.1.1. Assume that $\lambda_f(p^\alpha)\lambda_g(p^\alpha) \geq 0$ for all $p^\alpha \leq x$ with $\alpha \leq 2$. By Proposition 3.1.7, we see that

$$\sum_{\substack{n\leq x/2,\\(n,N)=1,\\n \text{ square-free}}} \lambda_f(n)\lambda_g(n)\log^2(x/n) \gg \sum_{\substack{n\leq x/2,\\(n,N)=1,\\n \text{ square-free}}} \lambda_f(n)\lambda_g(n) \gg \frac{x}{\log^2 x} \tag{3.1.7}$$

provided $x \geq \exp(c\log^2(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)}))$, where $c > 0$, $\mathfrak{q}(f)$ and $\mathfrak{q}(g)$ are as in Proposition 3.1.7. Now comparing (3.1.2) and (3.1.7), for any $\epsilon > 0$, we have

$$x \ll_\epsilon \max\left\{\exp\Big(c\log^2(\sqrt{\mathfrak{q}(f)} + \sqrt{\mathfrak{q}(g)})\Big), \left[N^2\left(1 + \frac{k_2 - k_1}{2}\right)\left(\frac{k_1 + k_2}{2}\right)\right]^{1+\epsilon}\right\}.$$

Here we have used Lemma 4 of Choie and Kohnen [13]. This completes the proof of Theorem 3.1.1. □

3.2 Sign changes in short intervals

In this section, we investigate sign changes of the sequence $\{a_f(n)a_g(n^2)\}_{n\in\mathbb{N}}$ in short intervals. The question of sign changes of the sequence $\{a_f(n)a_g(n)\}_{n\in\mathbb{N}}$ in short intervals was studied by Kumari and R. Murty (see [45, Theorem 1.6]). In fact, they proved the following theorem.

Theorem 3.2.1. *For any integer $k \geq 2$, let*

$$f(z) \;=\; \sum_{n=1}^{\infty} a_f(n)q^n \in S_k^{new}(N) \quad \textit{and} \quad g(z) \;=\; \sum_{n=1}^{\infty} a_g(n)q^n \in S_k^{new}(N)$$

be two distinct normalized Hecke eigenforms. For any sufficiently large x and any $\delta > 7/8$, the sequence $\{a_f(n)a_g(n)\}_{n\in\mathbb{N}}$ has at least one sign change in $(x, x + x^{\delta}]$. In particular, the number of sign changes for $n \leq x$ is $\gg x^{1-\delta}$.

Here we investigate sign changes of the sequence $\{a_f(n)a_g(n^2)\}_{n\in\mathbb{N}}$ in short intervals. In particular, we prove the following theorem.

Theorem 3.2.2. *For integers $k_1, k_2 \geq 2$, let*

$$f(z) \;=\; \sum_{n=1}^{\infty} a_f(n)q^n \in S_{k_1}(1) \quad \textit{and} \quad g(z) \;=\; \sum_{n=1}^{\infty} a_g(n)q^n \in S_{k_2}(1)$$

be two distinct normalized Hecke eigenforms. For any sufficiently large x and any $\delta > 17/18$, the sequence $\{a_f(n)a_g(n^2)\}_{n\in\mathbb{N}}$ changes sign at least once in $(x, x + x^{\delta}]$. In particular, the number of sign changes for $n \leq x$ is $\gg x^{1-\delta}$.

3.2.1 A lemma

We shall use the following lemma to prove Theorem 3.2.2.

Lemma 3.2.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_m\}_{m\in\mathbb{N}}$ be two sequences of real numbers such that*

1. $a_n = O(n^{\alpha_1}), \quad b_m = O(m^{\alpha_2})$,
2. $\sum_{n,m\le x} a_n b_m \ll x^{\beta}$,
3. $\sum_{n,m\le x} a_n^2 b_m^2 = cx + O(x^{\gamma})$,

where $\alpha_1, \alpha_2, \beta, \gamma \ge 0$ and $c > 0$ such that $\max\{\alpha_1+\alpha_2+\beta, \gamma\} < 1$. Then for any r satisfying

$$\max\{\alpha_1+\alpha_2+\beta, \gamma\} < r < 1,$$

there exists a sign change among the elements of the sequence $\{a_n b_m\}_{n,m\in\mathbb{N}}$ for any $n, m \in [x, x+x^r]$. Consequently, for sufficiently large x, the number of sign changes among the elements of the sequence $\{a_n b_m\}_{n,m\in\mathbb{N}}$ with $n, m \le x$ are $\gg x^{1-r}$.

Proof of Lemma 3.2.3. Suppose that for any $r \in \mathbb{R}$ satisfying

$$\max\{\alpha_1+\alpha_2+\beta, \gamma\} < r < 1,$$

the elements of the sequence $\{a_n b_m\}_{n,m\in\mathbb{N}}$ have same signs in $[x, x+x^r]$. This implies that

$$x^r \ll \sum_{x\le n,m\le x+x^r} a_n^2 b_m^2 \ll x^{\alpha_1+\alpha_2} \sum_{x\le n,m\le x+x^r} a_n b_m \ll x^{\alpha_1+\alpha_2+\beta},$$

which is a contradiction. This completes the proof of the lemma. □

Lemma 3.2.3 can be thought of as a generalization of a theorem of Meher and R. Murty (see [60, Theorem 1.1]) when $b_1 = 1$ and $b_m = 0$ for all $m > 1$.

3.2.2 Proof of Theorem 3.2.2

We shall apply Lemma 3.2.3 to prove Theorem 3.2.2. In order to apply Lemma 3.2.3, we need to verify following conditions for elements of the sequence $\{\lambda_f(n)\lambda_g(n^2)\}_n$. Note that

1. by Ramanujan-Petersson bound, for any $\epsilon > 0$ and any $n \in \mathbb{N}$, we have

$$\lambda_f(n)\lambda_g(n^2) \;=\; O_\epsilon(n^\epsilon).$$

2. by a recent work of Lü [53, Theorem 1.2(2)], one has

$$\sum_{n\leq x} \lambda_f(n)\lambda_g(n^2) \;\ll\; x^{5/7}(\log x)^{-\theta/2},$$

   where $\theta = 1 - \frac{8}{3\pi} = 0.1512\ldots$

3. in the same paper, Lü (see [53, Lemma 2.3(iii)]) also proved that

$$\sum_{n\leq x} \lambda_f(n)^2\lambda_g(n^2)^2 \;=\; cx \;+\; O(x^{\frac{17}{18}+\epsilon}),$$

   where $c > 0$.

Now Theorem 3.2.2 follows from Lemma 3.2.3 by choosing $a_n = \lambda_f(n)$ and $b_m = \lambda_g(m^2)$ for all $m, n \in \mathbb{N}$ and considering the sequence $\{a_n b_n\}_{n\in\mathbb{N}}$.

3.3 Hecke eigenvalues of non-CM forms

In this section, we briefly review some of the recent results about distribution of the Hecke eigenvalues of newforms which are not of CM type and establish some of its variants. We shall use these results to prove our theorems in upcoming sections.

One of the deepest result in the theory of non-CM forms is the famous conjecture predicted independently by Sato and Tate about equidistribution of Hecke eigenvalues of a non-CM form. This is now a theorem due to Barnet-Lamb, Geraghty, Harris and Taylor (see [10, Theorem B(3)]). Before stating their theorem, let us introduce the notion of equidistribution / uniform distribution.

Definition 3.3.1. *Let $\tilde{\mu}$ be a non-negative regular Borel measure on a compact Hausdorff space X such that $\tilde{\mu}(X) = 1$. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is said to be $\tilde{\mu}$-equidistributed or $\tilde{\mu}$-uniformly distributed in X if for any continuous function $f : X \to \mathbb{R}$, one has*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{1\leq n\leq N} f(x_n) \;=\; \int_X f \; d\tilde{\mu}.$$

In the special case, when $X := [0,1]$ and $\tilde{\mu}$ is the usual Lebesgue measure, one has the following equivalent criterion due to Weyl.

Theorem 3.3.2. *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is uniformly distributed in $[0,1]$ if and only if*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} \;=\; 0 \quad \text{for all integers} \;\; m \neq 0.$$

For a proof of Weyl's criterion on uniform distribution, see page 7 of Kuipers and Niederreiter [44]. One can also see R. Murty [66].

Now we are in a position to state the Sato-Tate theorem.

Theorem 3.3.3. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform which is a non-CM form. If we write $\lambda_f(p) = 2\cos\theta_f(p)$ with $\theta_f(p) \in [0,\pi]$, then $\theta_f(p)$ is μ_{ST}-uniformly distributed in $[0,\pi]$, where $\mu_{ST} := (2/\pi)\sin^2\theta d\theta$ is the Sato-Tate measure.*

Another important result in this direction is the following theorem of R. Murty and Pujahari [69, Theorem 1.1] (also see Patankar and Rajan [72]).

Theorem 3.3.4. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms. If at least one of f or g is a non-CM form and*

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \lambda_f(p) = \lambda_g(p)\}}{x/\log x} \;>\; 0,$$

then $f = g \otimes \chi$ for some Dirichlet character χ.

As an immediate corollary, we have the following.

Corollary 3.3.5. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be as in Theorem 3.3.4. Moreover, if $f \neq g \otimes \chi$ for any Dirichlet character χ, then the natural density of the set*

$$\{p \in \mathcal{P} \mid \lambda_f(p) \;\neq\; \lambda_g(p)\}$$

exists and is equal to one.

Here we have the following variant of Theorem 3.3.4.

Theorem 3.3.6. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms. Also let at least one of f or g be of non-CM type and write*

$$\lambda_f(p) \;:=\; 2\cos\theta_f(p) \quad \text{and} \quad \lambda_g(p) \;:=\; 2\cos\theta_g(p)$$

with $\theta_f(p), \theta_g(p) \in [0, \pi]$. For any fixed $\alpha \in [-\pi, \pi]$, if one has

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \theta_f(p) \;-\; \theta_g(p) \;=\; \alpha\}}{x/\log x} \;>\; 0,$$

then $f = g \otimes \chi$ for some Dirichlet character χ.

In order to prove this theorem, we follow the course of action adopted in [69]. We start with the following propositions.

Proposition 3.3.7. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms, not both CM. If $f \neq g \otimes \chi$ for any Dirichlet character χ, then for any $m, n \in \mathbb{N}$, one has*

$$\sum_{p \leq x} \frac{\sin[(m+1)\theta_f(p)] \sin[(n+1)\theta_g(p)]}{\sin \theta_f(p) \sin \theta_g(p)} \;=\; o\left(\frac{x}{\log x}\right)$$

as x tends to infinity.

For a proof of this proposition see [32, 69]. Before we state the next proposition, we need to introduce the following function;

Definition 3.3.8. *For any $0 < \delta < \pi$, let $f_\delta : [-\pi, \pi] \to \mathbb{R}$ be defined by*

$$f_\delta(x) \;:=\; \begin{cases} 1 - \frac{|x|}{\delta} & \text{if } |x| \leq \delta; \\ 0 & \text{otherwise.} \end{cases}$$

We then extend the function f_δ to whole of $\mathbb{R}$ as a periodic function with period 2π.

Proposition 3.3.9. *Let f_δ be as in Definition 3.3.8. Then for any natural number M, we have*

$$f_\delta(x) \;=\; \frac{\delta}{2\pi} \;+\; 2\sum_{n=1}^{M} \frac{1-\cos[n\delta]}{\pi n^2 \delta} \cos[nx] \;+\; O\left(\frac{1}{M\delta}\right),$$

where the implied constant is absolute.

Proof of this proposition can be found in [69].

Proof of Theorem 3.3.6. First suppose that at least one of $\theta_f(p), \theta_g(p)$ is either 0 or π. In this case the other one takes the value $\pm\alpha$ or $\pi \pm \alpha$. Applying Theorem 3.3.3 in these cases, we get

$$\#\{p \leq x \mid \theta_f(p) - \theta_g(p) = \alpha\} \;=\; o\left(\frac{x}{\log x}\right).$$

Thus from now onwards we can assume that neither $\theta_f(p)$ nor $\theta_g(p)$ take the values 0 or π. Using the definition of the function f_δ, one can write

$$\begin{aligned}\#\left\{p \le x \mid \theta_f(p) - \theta_g(p) \;=\; \alpha\right\} \;\le\; & \sum_{p\le x} f_\delta\big(\theta_f(p) - \theta_g(p) - \alpha\big) \\ & + \sum_{p\le x} f_\delta\big(\theta_f(p) + \theta_g(p) + \alpha\big) \;+\; \sum_{p\le x} f_\delta\big(\theta_f(p) - \theta_g(p) + \alpha\big) \\ & + \sum_{p\le x} f_\delta\big(\theta_f(p) + \theta_g(p) - \alpha\big).\end{aligned}$$

Now by applying Proposition 3.3.9, the right hand side of the above inequality can be written as

$$\frac{2\delta\pi(x)}{\pi} \;+\; 8\sum_{n=1}^{M}\frac{1-\cos[n\delta]}{\pi n^2\delta}\cos[n\alpha]\sum_{p\le x}\cos[n\theta_f(p)]\cos[n\theta_g(p)] \;+\; O\left(\frac{\pi(x)}{M\delta}\right),$$

where

$$\pi(x) := \#\left\{p \in \mathcal{P} \mid p \le x\right\}.$$

When $n = 1$, by the theory of Rankin-Selberg L-functions, one has

$$\sum_{p\le x}\cos\theta_f(p)\cos\theta_g(p) \;=\; o\big(\pi(x)\big) \tag{3.3.1}$$

as x tends to infinity. For $n \ge 2$, one can write

$$\begin{aligned}4\cos[n\theta_f(p)]\cos[n\theta_g(p)] \;=\; & \left(\frac{\sin[(n+1)\theta_f(p)]}{\sin\theta_f(p)} - \frac{\sin[(n-1)\theta_f(p)]}{\sin\theta_f(p)}\right) \\ & \times\left(\frac{\sin[(n+1)\theta_g(p)]}{\sin\theta_g(p)} - \frac{\sin[(n-1)\theta_g(p)]}{\sin\theta_g(p)}\right).\end{aligned}$$

Now suppose that $f \neq g \otimes \chi$ for any Dirichlet character χ. Then by using (3.3.1) and Proposition 3.3.7, we get

$$\sum_{p\le x}\cos[n\theta_f(p)]\cos[n\theta_g(p)] \;=\; o\big(\pi(x)\big)$$

as x tends to infinity. Thus

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \theta_f(p) - \theta_g(p) = \alpha\}}{\pi(x)} \;\le\; \frac{2\delta}{\pi} \;+\; O\left(\frac{1}{M\delta}\right).$$

As M tends to infinity, we get

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \theta_f(p) - \theta_g(p) = \alpha\}}{\pi(x)} \;\le\; \frac{2\delta}{\pi}.$$

Since δ can be chosen arbitrarily small, we have

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \theta_f(p) - \theta_g(p) = \alpha\}}{\pi(x)} \;=\; 0,$$

as required. This completes the proof of Theorem 3.3.6.

We shall need another variant of Theorem 3.3.4 which is stated below.

Theorem 3.3.10. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms. Also assume that at least one of f or g is of non-CM type and*

$$\lambda_f(p) \;=\; 2\cos\theta_f(p) \quad \textit{and} \quad \lambda_g(p) \;=\; 2\cos\theta_g(p)$$

with $\theta_f(p), \theta_g(p) \in [0, \pi]$. For any fixed $\alpha \in [0, 2\pi]$, if one has

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \theta_f(p) + \theta_g(p) = \alpha\}}{x/\log x} \;>\; 0,$$

then $f = g \otimes \chi$ for some Dirichlet character χ.

The proof follows exactly along the lines of the proof of Theorem 3.3.6 and hence we omit the proof here.

3.4 Hecke eigenvalues of CM Forms

In this section, we shall recall various well-known properties of Hecke eigenvalues of newforms which are CM forms. We use these properties to study joint equidistribution of non-zero Hecke eigenvalues of two distinct newforms which are of CM type. These joint equidistribution results might be known to the experts but we could not find any reference in the literature. Hence we investigate these properties here as we shall need them to prove our theorems in the next section. We shall start with some notations and definitions.

Let K be an imaginary quadratic field and $\mathcal{O}_K$ be its ring of integers. For any ideal $\mathfrak{M} \subset \mathcal{O}_K$, let $I(\mathfrak{M})$ be the set of fractional ideals of K which are coprime to $\mathfrak{M}$ and

$$P(\mathfrak{M}) \ := \ \left\{(a) \in I(\mathfrak{M}) \mid a \in K^{\times},\ a \equiv 1 \mod{}^{\times}\, \mathfrak{M}\right\},$$

where $a \equiv 1 \mod{}^{\times} \mathfrak{M}$ implies that $v_{\mathfrak{P}}(a-1) \geq v_{\mathfrak{P}}(\mathfrak{M})$ for all $\mathfrak{P} \mid \mathfrak{M}$. Note that $P(\mathfrak{M})$ is a subgroup of $I(\mathfrak{M})$.

Definition 3.4.1. *A character $\chi : I(\mathfrak{M}) \to S^1$ is called a Hecke character mod $\mathfrak{M}$ if for all $(a) \in P(\mathfrak{M})$, we have*

$$\chi((a)) \ = \ \left(\frac{a}{|a|}\right)^{u}, \tag{3.4.1}$$

where $u \in \mathbb{Z}$. Further, a Hecke character χ mod $\mathfrak{M}$ *is called primitive if it is not a Hecke character for any $\mathfrak{N} \mid \mathfrak{M}$. In this case, we call $\mathfrak{M}$ to be the conductor of χ.*

Remark 3.4.2. *If the integer u in Definition 3.4.1 is non-zero, then the Hecke character χ is not of finite order, that is, there does not exist any integer $m \in \mathbb{Z}\backslash\{0\}$ such that χ^m is the trivial Hecke character. As otherwise, for all $a \in K^{\times}$ with $a \equiv 1$* mod ${}^{\times}\mathfrak{M}$, *we have*

$$\left(\frac{a}{|a|}\right)^{mu} = 1.$$

This implies that the set $\{a \in K^\times \mid a \equiv 1 \mod {}^\times\mathfrak{M}\}$ *is contained in the union of finitely many lines in* $\mathbb{C}^\times$. *This is not true as by approximation theorem, the set* $\{a \in K^\times \mid a \equiv 1 \mod {}^\times\mathfrak{M}\}$ *is dense in* $\mathbb{C}^\times$.

In this set-up, one has the following characterization theorem of CM forms.

Theorem 3.4.3. *Let* $K = \mathbb{Q}(\sqrt{d})$ *be an imaginary quadratic field with discriminant* d. *Also let* $\mathfrak{M}$ *be an integral ideal of* K *and* $\chi \mod \mathfrak{M}$ *be a Hecke character with* u *as in* (3.4.1) *is assumed to be positive. For* $z \in \mathcal{H}$, *define*

$$f(z) := \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N(\mathfrak{a})^{u/2} q^{N(\mathfrak{a})}, \tag{3.4.2}$$

where $\mathfrak{a}$ *varies over ideals of* $\mathcal{O}_K$ *which are coprime to* $\mathfrak{M}$ *and* $N(\mathfrak{a})$ *denotes the absolute norm of* $\mathfrak{a}$. *Then* $f \in S_{u+1}(M, \tilde{\chi})$, *where* $M := |d|N(\mathfrak{M})$ *and* $\tilde{\chi}$ *is a Dirichlet character defined as follows:*

$$\tilde{\chi}(m) := \left(\frac{d}{m}\right) \chi((m))\ sgn(m)^u, \quad \text{for all } m \in \mathbb{Z}.$$

Moreover $f \in S^{new}_{u+1}(M, \tilde{\chi})$ *is a normalized Hecke eigenform of CM type if* χ *is a primitive character* $\mod \mathfrak{M}$. *Conversely, any* $f \in S^{new}_{u+1}(M, \tilde{\chi})$, *which is a Hecke eigenform of CM type, is obtained from a primitive Hecke character of an imaginary quadratic field in this way.*

For a proof of above theorem one can see [61, Theorem 4.8.2] and [82, Theorem 4.5].

From now on, we say that a newform f of CM type has CM by an imaginary quadratic field K if f comes from a primitive Hecke character of the field K in the sense of Theorem 3.4.3. It is clear from the identity (3.4.2) that for any prime p

with $(p, M) = 1$, we have the following:

$$\lambda_f(p) = \begin{cases} \chi(\mathfrak{P}) + \chi(\overline{\mathfrak{P}}) & \text{if } p\mathcal{O}_K = \mathfrak{P}\overline{\mathfrak{P}},\ \mathfrak{P} \neq \overline{\mathfrak{P}}; \\ 0 & \text{if } p\mathcal{O}_K \text{ is inert.} \end{cases}$$

The study of equidistribution of the values of a Hecke character goes back to Hecke [33] (see also Rajan [77]); these equidistribution results will imply the following equidistribution result for a CM newform which is a normalized Hecke eigenform.

Theorem 3.4.4. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform of CM type. Also let $I \subset [0, \pi]$ be such that $\pi/2 \notin I$ and I be a union of finitely many disjoint intervals. Now if we write $\lambda_f(p) = 2\cos\theta_f(p)$, then we have*

$$\lim_{x\to\infty} \frac{\#\{p \leq x \mid p \in \mathcal{P},\ \theta_f(p) \in I\}}{\#\{p \leq x \mid p \in \mathcal{P}\}} = \frac{|I|}{2\pi}.$$

Further, the set $\{p \in \mathcal{P} \mid \theta_f(p) = \pi/2\}$ has natural density $1/2$.

A proof of this theorem can be found in [5, Theorem 3.1.1]. In this set-up, we prove the following theorem;

Theorem 3.4.5. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms of CM type. If $k_1 \neq k_2$, then*

$$\limsup_{x\to\infty} \frac{\#\{p \leq x \mid \lambda_f(p) = \lambda_g(p)\}}{x/\log x} \leq \frac{1}{2}.$$

In order to prove this theorem, we need the following lemmas.

Lemma 3.4.6. *Let K be an imaginary quadratic field and $\mathfrak{M}, \mathfrak{N}$ be integral ideals of K. Also let $\chi \mod \mathfrak{M}$ and $\psi \mod \mathfrak{N}$ be primitive Hecke characters of K. Then the sequence*

$$\{(\chi(\mathfrak{P}), \psi(\mathfrak{P}))\}_{\mathfrak{P}},$$

where $\mathfrak{P}$ varies over all prime ideals of $\mathcal{O}_K$ with $(\mathfrak{P}, \mathfrak{M}\mathfrak{N}) = 1$ is equidistributed in $S^1 \times S^1$ if and only if $\chi^m \neq \psi^n$ for any $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

Proof of Lemma 3.4.6. For any prime ideal $\mathfrak{P}$ in $\mathcal{O}_K$ with $(\mathfrak{P}, \mathfrak{M}\mathfrak{N}) = 1$, let us write

$$\chi(\mathfrak{P}) \;=\; e^{2\pi i \theta_\chi(\mathfrak{P})} \quad \text{and} \quad \psi(\mathfrak{P}) \;=\; e^{2\pi i \theta_\psi(\mathfrak{P})}$$

with $-1/2 < \theta_\chi(\mathfrak{P}), \theta_\psi(\mathfrak{P}) \leq 1/2$. Then by Weyl's uniform distribution criterion, one knows that the sequence $\{(\theta_\chi(\mathfrak{P}), \theta_\psi(\mathfrak{P}))\}_{\mathfrak{P}}$ is equidistributed in $[-1/2, 1/2] \times [-1/2, 1/2]$ if and only if for any $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$

$$\sum_{\substack{N\mathfrak{P} \leq x \\ (\mathfrak{P}, \mathfrak{M}\mathfrak{N})=1}} e^{2\pi i (m\theta_\chi(\mathfrak{P}) + n\theta_\psi(\mathfrak{P}))} \;=\; o\big(\pi_K(x)\big) \iff \sum_{\substack{N\mathfrak{P} \leq x \\ (\mathfrak{P}, \mathfrak{M}\mathfrak{N})=1}} \chi(\mathfrak{P})^m \psi(\mathfrak{P})^n \;=\; o\big(\pi_K(x)\big),$$

where $\pi_K(x)$ denotes the number of prime ideals $\mathfrak{P}$ of K with $N\mathfrak{P} \leq x$.

For any complex number $s \in \mathbb{C}$ with $\Re(s) > 1$, let us consider the Hecke L-function

$$L(s, \chi^m \psi^n) := \sum_{a} \frac{\chi(\mathfrak{a})^m \psi(\mathfrak{a})^n}{N(\mathfrak{a})^s},$$

where the summation is over all non-zero integral ideals $\mathfrak{a}$ of K which are coprime to $\mathfrak{M}\mathfrak{N}$. This defines a holomorphic function in the region $\Re(s) > 1$. It is well known that the function $L(s, \chi^m \psi^n)$ can be analytically continued to the entire complex plane if and only if $\chi^m \psi^n$ is not the trivial character. When $\chi^m \psi^n$ is equal to the trivial character, then $L(s, \chi^m \psi^n)$ has a simple pole at $s = 1$. Further, $L(s, \chi^m \psi^n)$ is non-vanishing on the line $\Re(s) = 1$. Hence by the *Wiener-Ikehara Tauberian theorem* (see [67, page 7]) and the partial summation formula, one has

$$\sum_{\substack{N\mathfrak{P} \leq x \\ (\mathfrak{P}, \mathfrak{M}\mathfrak{N})=1}} \chi(\mathfrak{P})^m \psi(\mathfrak{P})^n \;=\; o\big(\pi_K(x)\big)$$

if and only if $\chi^m \psi^n$ is not the trivial character, that is, if $\chi^m \neq \psi^{-n}$. This completes the proof of Lemma 3.4.6. $\square$

We also need the following lemma to prove Theorem 3.4.5.

Lemma 3.4.7. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms of CM type. If*

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \lambda_f(p) = \lambda_g(p)\}}{x/\log x} > \frac{1}{2},$$

then both f and g have CM by the same field.

Proof of Lemma 3.4.7. Let f and g have CM by the fields K_f and K_g respectively. Also let $\psi_f \mod \mathfrak{M}$ and $\psi_g \mod \mathfrak{N}$ be primitive Hecke characters corresponding to f and g respectively. Then for any prime p with $(p, N_1N_2) = 1$, we have

$$\lambda_f(p) = \begin{cases} \psi_f(\mathfrak{P}) + \psi_f(\overline{\mathfrak{P}}) & \text{if } p\mathcal{O}_{K_f} = \mathfrak{P}\overline{\mathfrak{P}},\ \mathfrak{P} \neq \overline{\mathfrak{P}}; \\ 0 & \text{if } p\mathcal{O}_{K_f} \text{ is inert;} \end{cases}$$

and

$$\lambda_g(p) = \begin{cases} \psi_g(\mathfrak{q}) + \psi_g(\overline{\mathfrak{q}}) & \text{if } p\mathcal{O}_{K_g} = \mathfrak{q}\overline{\mathfrak{q}},\ \mathfrak{q} \neq \overline{\mathfrak{q}}; \\ 0 & \text{if } p\mathcal{O}_{K_g} \text{ is inert.} \end{cases}$$

Set $N := N_1N_2$ and

- $\mathcal{P}_N := \big\{p \in \mathcal{P} \mid (p, N) = 1\big\}$;
- $\mathcal{P}_{f,g} := \{p \in \mathcal{P}_N \mid \lambda_f(p) = \lambda_g(p)\}$;
- $\mathcal{P}_{s,s} := \{p \in \mathcal{P}_N \mid p \text{ splits both in } K_f \text{ and } K_g\}$;
- $\mathcal{P}_{s,n} := \{p \in \mathcal{P}_N \mid p \text{ splits in } K_f \text{ but not in } K_g\}$;
- $\mathcal{P}_{n,s} := \{p \in \mathcal{P}_N \mid p \text{ splits in } K_g \text{ but not in } K_f\}$;
- $\mathcal{P}_{n,n} := \{p \in \mathcal{P}_N \mid p \text{ does not split in both } K_f \text{ and } K_g\}$.

Now assume that $K_f \neq K_g$. Then by *Chebotarev density theorem*, natural density of each of the sets $P_{s,s}, P_{s,n}, P_{n,s}, P_{n,n}$ is $1/4$. Since both f and g are CM forms, we know that $\lambda_f(p) = 0$ (respectively $\lambda_g(p) = 0$) if and only if p is inert in the imaginary quadratic field K_f (respectively K_g) (see Serre [91, page 180]). Thus we have

$$P_{f,g} \cap P_{n,s} = \emptyset \quad \text{and} \quad P_{f,g} \cap P_{s,n} = \emptyset,$$

a contradiction to our hypothesis. This completes the proof of Lemma 3.4.7. $\square$

Proof of Theorem 3.4.5. Let $\chi \mod \mathfrak{M}$ and $\psi \mod \mathfrak{N}$ be the primitive Hecke characters corresponding to f and g respectively. Suppose that

$$\limsup_{x\to\infty} \frac{\#\{p \le x \mid \lambda_f(p) = \lambda_g(p)\}}{x/\log x} > \frac{1}{2}$$

as otherwise there is nothing to prove. Hence by applying Lemma 3.4.7, we can conclude that the CM forms f and g have CM by the same field, say K.

For any prime p with $(p, N_1N_2) = 1$, it follows from (3.4.2) that if $p\mathcal{O}_K = \mathfrak{P}\overline{\mathfrak{P}}$, $\mathfrak{P} \neq \overline{\mathfrak{P}}$, then

$$\lambda_f(p) = \chi(\mathfrak{P}) + \chi(\overline{\mathfrak{P}}) \quad \text{and} \quad \lambda_g(p) = \psi(\mathfrak{P}) + \psi(\overline{\mathfrak{P}})$$

and if p is inert, then $\lambda_f(p) = 0 = \lambda_g(p)$. For any prime ideal $\mathfrak{P}$ in $\mathcal{O}_K$ with $(\mathfrak{P}, \mathfrak{M}\mathfrak{N}) = 1$, let us write

$$\chi(\mathfrak{P}) = e^{2\pi i\theta_\chi(\mathfrak{P})} \quad \text{and} \quad \psi(\mathfrak{P}) = e^{2\pi i\theta_\psi(\mathfrak{P})}$$

with $-1/2 < \theta_\chi(\mathfrak{P}), \theta_\psi(\mathfrak{P}) \le 1/2$. Note that if $p\mathcal{O}_K = \mathfrak{P}\overline{\mathfrak{P}}$ with $\mathfrak{P} \neq \overline{\mathfrak{P}}$, then as f and g have trivial Nebentypus, we have

$$\chi(\overline{\mathfrak{P}}) = \overline{\chi(\mathfrak{P})} \quad \text{and} \quad \psi(\overline{\mathfrak{P}}) = \overline{\psi(\mathfrak{P})}. \tag{3.4.3}$$

Now by Lemma 3.4.6, we know that $\{(\theta_\chi(\mathfrak{P}),\ \theta_\psi(\mathfrak{P}))\}_{\mathfrak{P}}$ is equidistributed in $[-1/2, 1/2] \times [-1/2, 1/2]$ if and only if $\chi^m \neq \psi^n$ for any $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$.

First suppose that the sequence $\{(\theta_\chi(\mathfrak{P}),\ \theta_\psi(\mathfrak{P}))\}_{\mathfrak{P}}$ is equidistributed in $[-1/2, 1/2] \times [-1/2, 1/2]$. This implies that the set

$$\#\{N\mathfrak{P} \leq x \mid \theta_\chi(\mathfrak{P}) = \pm\theta_\psi(\mathfrak{P})\} \;=\; o\big(\pi_K(x)\big) \tag{3.4.4}$$

as x tends to infinity. Since

$$\begin{aligned}\{p \in \mathcal{P} \mid (p, N_1N_2) = 1,\ \lambda_f(p) = \lambda_g(p)\} &= \big\{p \in \mathcal{P} \mid (p, N_1N_2) = 1,\ p \text{ is inert in } K\big\} \\ &\cup\ \big\{p \in \mathcal{P} \mid (p, N_1N_2) = 1,\ p\mathcal{O}_K = \mathfrak{P}\overline{\mathfrak{P}},\ \mathfrak{P} \neq \overline{\mathfrak{P}},\ \theta_\chi(\mathfrak{P}) = \pm\theta_\psi(\mathfrak{P})\big\},\end{aligned}$$

then by equations (3.4.3), (3.4.4) and the fact that the set $\{p \in \mathcal{P} \mid p \text{ is inert in } K\}$ has natural density 1/2, we get

$$d\left(\{p \in \mathcal{P} \mid \lambda_f(p) = \lambda_g(p)\}\right) \;=\; \frac{1}{2},$$

which contradicts our assumption.

Now suppose that $\{(\theta_\chi(\mathfrak{P}), \theta_\psi(\mathfrak{P}))\}_{\mathfrak{P}}$ is not equidistributed in $[-1/2, 1/2] \times [-1/2, 1/2]$, then by Lemma 3.4.6, there exists a pair of integers $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $\chi^m = \psi^n$. Note that by Remark (3.4.2), neither χ nor ψ is of finite order and hence $m \neq 0$ as well as $n \neq 0$. Without loss of generality we can choose a pair of integers (m, n) with smallest $m > 0$ such that $\chi^m = \psi^n$. Since both χ and ψ correspond to CM forms, $m > 0$ implies that $n > 0$. Using binomial theorem and induction on $t \in \mathbb{N}$, one can write

$$\big(\chi(\mathfrak{P}) + \overline{\chi(\mathfrak{P})}\big)^t \;=\; \chi(\mathfrak{P})^t \;+\; \overline{\chi(\mathfrak{P})}^t \;+\; P_{t-2}\big(\lambda_f(p)\big),$$

where P_{t-2} is a polynomial of degree less than or equal to $t - 2$. This implies that there exist polynomials P_m and Q_n of degrees m and n respectively such that

$$P_m\big(\lambda_f(p)\big) \;=\; Q_n\big(\lambda_g(p)\big) \tag{3.4.5}$$

for all split primes p with $(p, N_1N_2) = 1$. If $\lambda_f(p) = \lambda_g(p) = \alpha$, say, then from equation (3.4.5), it follows that α is a root of the polynomial $P_m - Q_n$.

Now if $m \neq n$, then the polynomial $P_m - Q_n$ is a non-zero polynomial. Note that $\alpha \neq 0$ as p splits in K. Applying Theorem 3.4.4, we know that the set of primes p for which $\lambda_f(p) = \alpha'$ for any $\alpha' \neq 0$ has density zero. Since α can take at most finitely many values, we have

$$d\left(\{p \in \mathcal{P} \mid p \text{ splits in } K,\ \lambda_f(p) = \lambda_g(p)\}\right) \;=\; 0.$$

When $m = n$, then $\chi\psi^{-1}$ has finite order. This is a contradiction to Remark (3.4.2) as by hypothesis, we have $k_1 \neq k_2$.

The above observations along with the fact that

$$d\left(\{p \in \mathcal{P} \mid \lambda_f(p) = \lambda_g(p) = 0\}\right) \;=\; \frac{1}{2}$$

implies that the set $\{p \in \mathcal{P} \mid \lambda_f(p) = \lambda_g(p)\}$ has natural density $1/2$, a contradiction to our assumption. This completes the proof of Theorem 3.4.5. $\square$

3.5 Sign change and multiplicity one theorem

In this section, we first relate the question of simultaneous sign change of Hecke eigenvalues to that of multiplicity one theorem. Then we use this to prove some quantitative sign change results of Hecke eigenvalues.

Theorem 3.5.1. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigen-*

forms and p be a prime such that $(p, N_1N_2) = 1$. Then the following conditions are equivalent;

1. *there exist infinitely many $m \geq 1$ such that $\lambda_f(p^m)\lambda_g(p^m) > 0$ and infinitely many $m \geq 1$ such that $\lambda_f(p^m)\lambda_g(p^m) < 0$;*

2. *one has $\lambda_f(p) \neq \lambda_g(p)$.*

The above theorem then enables us to estimate the density of primes p for which the sequence $\{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ changes sign infinitely often. In particular, we prove the following theorem.

Theorem 3.5.2. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms and S be the set of primes p for which the sets*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) > 0\} \quad \textit{and} \quad \{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) < 0\}$$

are infinite. Then the density of the set S is as follows:

1. *if at least one of f or g is a non-CM form, then*
    - *the natural density of S is 1 provided $f \neq g \otimes \chi$ for any Dirichlet character χ;*
    - *the natural density of S is $1/2$ if $f = g \otimes \chi$ for some Dirichlet character χ.*

2. *if both f, g are of CM type, then*
    - *the lower natural density of S is greater than or equal to $1/2$ provided $k_1 \neq k_2$ or f, g have CM by different fields;*
    - *the lower natural density of S is greater than or equal to $1/8$ provided $k_1 = k_2$ and both f and g have CM by the same field.*

Theorem 3.5.2 improves a recent result of Gun, Kohnen and Rath [27, Theorem 3]. If we assume that at least one of f or g is non-CM form, then we can prove the following stronger result.

Theorem 3.5.3. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be distinct normalized Hecke eigenforms and at least one of f or g be not of CM type. For any positive integer $j \geq 1$, let S_j be the set of primes p such that*

$$\big\{m \in \mathbb{N} \mid a_f(p^{jm})a_g(p^{jm}) > 0\big\} \quad \text{and} \quad \big\{m \in \mathbb{N} \mid a_f(p^{jm})a_g(p^{jm}) < 0\big\}$$

are infinite. Then the natural density $d(S_j)$ of the set S_j is as follows;

1. *if $f \neq g \otimes \chi$ for any Dirichlet character χ, then $d(S_j) = 1$ for any $j \in \mathbb{N}$.*
2. *when $f = g \otimes \chi$ for some Dirichlet character χ, then*
   - *$d(S_j) = 1/2$ if j is odd;*
   - *$d(S_j) = 0$ otherwise.*

The above theorem can be thought of as a generalization of the following result of Kohnen and Martin [40].

Theorem 3.5.4. *Let $f \in S_k(1)$ be a normalized Hecke eigenform. Then for any integer $j \geq 1$ and for almost all primes p, the sequence $\{a_f(p^{nj})\}_{n\in\mathbb{N}}$ changes sign infinitely often.*

As mentioned in Remark 3.1 of [17], the proof of above theorem does not work for natural numbers j with $4 \mid j$. Here we prove the following theorem.

Theorem 3.5.5. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform and $j \geq 1$ be an integer. Consider the set S_j of primes p for which the sets*

$$\big\{m \in \mathbb{N} \mid a_f(p^{jm}) > 0\big\} \quad \text{and} \quad \big\{m \in \mathbb{N} \mid a_f(p^{jm}) < 0\big\}$$

are infinite. Then

1. *if f is a non-CM form, then the natural density of S_j is* 1*;*

2. *if f is of CM type and*

   - $4 \mid j$*, then the natural density of S_j is* $1/2$*;*

   - $4 \nmid j$*, then the natural density of S_j is* 1*.*

To prove the above theorem, we need the following one.

Theorem 3.5.6. *Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform and $j \geq 1$ be a natural number. Then for almost all primes p, the following conditions are equivalent;*

1. *there exists infinitely many integers $m \geq 1$ such that $\lambda_f(p^{jm}) > 0$ and infinitely many natural numbers $m \geq 1$ such that $\lambda_f(p^{jm}) < 0$;*

2. *one has*

$$\lambda_f(p) \notin \begin{cases} \{2\} & \text{for } j \text{ is odd;} \\ \{2, -2\} & \text{for } j \equiv 2 \ (mod\ 4); \\ \{-2, 0, 2\} & \text{for } j \equiv 0 \ (mod\ 4). \end{cases}$$

Furthermore, when $k \geq 4$ or $j = 1$, then the above equivalence is true for all primes p with $(p, N) = 1$.

3.5.1 Proof of Theorem 3.5.1

Recall that for any prime p with $(p,\ N_1N_2) = 1$, one can write

$$\lambda_f(p) \ := \ 2\cos\theta_f(p) \quad \text{and} \quad \lambda_g(p) \ := \ 2\cos\theta_g(p),$$

where $\theta_f(p), \theta_g(p) \in [0, \pi]$. Then for any integer $m \geq 1$, we have

$$\lambda_f(p^m) = \begin{cases} m+1 & \text{if } \theta_f(p) = 0; \\ (-1)^m(m+1) & \text{if } \theta_f(p) = \pi; \\ \frac{\sin[(m+1)\theta_f(p)]}{\sin\theta_f(p)} & \text{otherwise;} \end{cases}$$

and

$$\lambda_g(p^m) = \begin{cases} m+1 & \text{if } \ \theta_g(p) = 0; \\ (-1)^m(m+1) & \text{if } \ \theta_g(p) = \pi; \\ \frac{\sin[(m+1)\theta_g(p)]}{\sin\theta_g(p)} & \text{otherwise.} \end{cases}$$

Let us first assume that $\lambda_f(p) = \lambda_g(p)$. Then for any $m \geq 1$, we have $\lambda_f(p^m) = \lambda_g(p^m)$ and hence $\lambda_f(p^m)\lambda_g(p^m) \geq 0$ for all $m \in \mathbb{N}$.

Now we assume that $\lambda_f(p) \neq \lambda_g(p)$, that is, $\theta_f(p) \neq \theta_g(p)$. Further if

$$\theta_f(p),\ \theta_g(p) \in \{0, \pi\},$$

then $\lambda_f(p^m)\lambda_g(p^m)$ is equal to $(-1)^m(m+1)^2$ which changes sign infinitely often.

Now suppose that at least one of $\theta_f(p), \theta_g(p)$ is equal to 0 or π and the other one lies in $(0, \pi)$. Without loss of generality, we assume that $\theta_f(p) = 0$ or π and $\theta_g(p) \in (0, \pi)$. Then for any integer $m \geq 1$, we have

$$\lambda_f(p^m)\lambda_g(p^m) = \begin{cases} \frac{(m+1)\sin[(m+1)\theta_g(p)]}{\sin\theta_g(p)} & \text{if } \ \theta_f(p) = 0 \ \text{ and } \ \theta_g(p) \in (0, \pi); \\ \frac{(m+1)\sin[(m+1)(\pi-\theta_g(p))]}{\sin[\pi-\theta_g(p)]} & \text{if } \ \theta_f(p) = \pi \ \text{ and } \ \theta_g(p) \in (0, \pi). \end{cases}$$

Since both $\theta_g(p)$ and $\pi - \theta_g(p)$ lie in $(0, \pi)$, the functions $\sin[m\theta_g(p)]$ and $\sin[m(\pi - \theta_g(p))]$ change sign infinitely often as m varies.

Finally assume that both $\theta_f(p), \theta_g(p) \in (0, \pi)$. In this case, we can write

$$\frac{\sin[m\theta_f(p)]\sin[m\theta_g(p)]}{\sin\theta_f(p)\sin\theta_g(p)} = \frac{\cos mu - \cos mv}{2\sin\theta_f(p)\sin\theta_g(p)},$$

where $u := \theta_f(p) - \theta_g(p),\ v := \theta_f(p) + \theta_g(p)$. Note that

$$u \not\equiv 0 \mod 2\pi, \quad v \not\equiv 0 \mod 2\pi, \quad u \not\equiv v \mod 2\pi, \quad u \not\equiv -v \mod 2\pi$$

as $\theta_f(p) \neq \theta_g(p)$ and $\theta_f(p), \theta_g(p) \in (0, \pi)$. We are done if we show that $(\cos mu - \cos mv)$ changes sign infinitely often as $m \in \mathbb{N}$ varies. On the contrary, we assume that it does not change sign infinitely often. Then there exists $m_0 \in \mathbb{N}$ such that

$$\big(\cos mu - \cos mv\big) \ \geq\ 0 \quad \text{or} \quad \big(\cos mu - \cos mv\big) \ \leq\ 0 \quad \text{for all} \quad m \geq m_0.$$

First assume that

$$\big(\cos mu - \cos mv\big) \ \geq\ 0 \tag{3.5.1}$$

for all $m \geq m_0$. Then for any $m' \geq m_0$, one has

$$\sum_{m=m_0}^{m'} \big(\cos mu - \cos mv\big)^2 \ \leq\ 2 \sum_{m=m_0}^{m'} \big(\cos mu - \cos mv\big).$$

The above identity along with (3.5.1) implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos mu - \cos mv\big)^2 \ \leq\ \lim_{N\to\infty} \frac{2}{N} \sum_{m=1}^{N} \big(\cos mu - \cos mv\big).$$

But on the other hand, we will show that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos mu - \cos mv\big)^2 \ \geq\ 1 \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos mu - \cos mv\big) \ =\ 0,$$

a contradiction. Indeed, for any real $x \not\equiv 0 \mod 2\pi$, using the identity

$$\frac{1}{2} \;+\; \cos x \;+\; \cos 2x \;+\; \cdots \;+\; \cos nx \;=\; \frac{\sin(n+\frac{1}{2})x}{2\sin(\frac{x}{2})},$$

one finds that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} \cos mx \;=\; \lim_{N\to\infty} \frac{1}{N}\left(\frac{\sin(N+\frac{1}{2})x}{2\sin(\frac{x}{2})} \;-\; \frac{1}{2}\right) \;=\; 0. \tag{3.5.2}$$

Since $u \not\equiv 0 \mod 2\pi$ and $v \not\equiv 0 \mod 2\pi$, using the identity (3.5.2), we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} \big(\cos mu \;-\; \cos mv\big) \;=\; 0.$$

Note that

$$\begin{aligned} &\lim_{N\to\infty} \sum_{m=1}^{N} \frac{(\cos mu \;-\; \cos mv)^2}{N} \\ =\;& \lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} \big(\cos^2 mu \;+\; \cos^2 mv\big) \\ -\;& \lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} \big(\cos[m(u+v)] \;+\; \cos[m(u-v)]\big). \end{aligned}$$

Using the facts $u \not\equiv v \mod 2\pi$ and $u \not\equiv -v \mod 2\pi$ and the identity (3.5.2), we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} \big(\cos[m(u+v)] \;+\; \cos[m(u-v)]\big) \;=\; 0.$$

Also we have

(3.5.3)

$$\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N} \big(\cos^2 mu \;+\; \cos^2 mv\big) \;=\; 1 \;+\; \lim_{N\to\infty} \frac{1}{2N}\sum_{m=1}^{N} \big(\cos 2mu \;+\; \cos 2mv\big).$$

By our assumptions, we have $2u \not\equiv 0 \mod 2\pi$. Also note that $2v \equiv 0 \mod 2\pi$ if and only if $v = \pi$. Thus the summation in the left hand side of the identity (3.5.3) is equal to 1 or 3/2 depending on $v \neq \pi$ or $v = \pi$. This completes the proof under assumption (3.5.1). Now if

$$(3.5.4) \qquad \big(\cos mu - \cos mv\big) \;\le\; 0$$

for all $m \geq m_0$, then consider the identity

$$\sum_{m=m_0}^{m'} \big(\cos mv - \cos mu\big)^2 \;\le\; 2\sum_{m=m_0}^{m'} \big(\cos mv - \cos mu\big).$$

Proceeding as in the previous case, this leads to a contradiction. This completes the proof of Theorem 3.5.1.

3.5.2 An elementary approach to strong multiplicity one theorem

We use the *Rankin-Selberg L-functions* associated to f and g to deduce the following proposition which is required to complete the proof of Theorem 3.5.2.

Proposition 3.5.7. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms. If $f \neq g$, then*

$$\liminf_{x\to\infty} \frac{\#\{p \le x \mid \lambda_f(p) \neq \lambda_g(p)\}}{x/\log x} \;\ge\; \frac{1}{8}.$$

Proof of Proposition 3.5.7. Let us define

$$\tilde{S} \;:=\; \{p \in \mathcal{P} \mid (p, N_1N_2) = 1 \text{ and } \lambda_f(p) \neq \lambda_g(p)\}.$$

Consider the sum

$$T(x): \;=\; \sum_{\substack{p\le x,\\ p\in\tilde{S}}} (\lambda_f(p) - \lambda_g(p))\, \lambda_f(p).$$

Note that

$$
\begin{aligned}
T(x) &= \sum_{\substack{p \leq x, \\ (p, N_1 N_2)=1}} \left(\lambda_f(p) - \lambda_g(p)\right) \lambda_f(p) \\
&= \sum_{\substack{p \leq x, \\ (p, N_1 N_2)=1}} \lambda_f(p)^2 - \sum_{\substack{p \leq x, \\ (p, N_1 N_2)=1}} \lambda_f(p) \lambda_g(p).
\end{aligned}
$$

It is well known that

$$
\sum_{\substack{p \leq x, \\ (p, N_1 N_2)=1}} \lambda_f(p)^2 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \quad \text{and} \quad \sum_{\substack{p \leq x, \\ (p, N_1 N_2)=1}} \lambda_g(p) \lambda_f(p) = o\left(\frac{x}{\log x}\right),
$$

as $x \to \infty$. Thus we have

$$
T(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right). \tag{3.5.5}
$$

Now if we put

$$
P^+ := \{p \in \mathcal{P} \mid \left(\lambda_f(p) - \lambda_g(p)\right) \lambda_f(p) > 0\},
$$

then we have

$$
T(x) \leq \sum_{\substack{p \leq x, \\ p \in P^+ \cap \tilde{S}}} \left(\lambda_f(p) - \lambda_g(p)\right) \lambda_f(p) \leq 8 \, \# \left\{p \leq x \mid p \in P^+ \cap \tilde{S}\right\}, \tag{3.5.6}
$$

where the last inequality follows from the fact that $|\lambda_f(p)| \leq 2$. Combining equations (3.5.5) and (3.5.6), we now have

$$
\# \left\{p \leq x \mid p \in \tilde{S}\right\} \geq \frac{x}{8 \log x} + o\left(\frac{x}{\log x}\right).
$$

This proves that the lower natural density of the set $\tilde{S}$ is greater than or equal to 1/8. $\square$

Note that this gives an elementary proof of a result of Ramakrishnan [78, Theorem 1] in this context.

3.5.3 Proof of Theorem 3.5.2

In this subsection, we shall provide a proof of Theorem 3.5.2. Note that sign changes of the sequence $\{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ is equivalent to the sign changes of the sequence $\{\lambda_f(p^m)\lambda_f(p^m)\}_{m\in\mathbb{N}}$. Using Theorem 3.5.1, we see that the set S contains a subset $\tilde{S}$ defined by

$$\tilde{S} \ := \ \{p \in \mathcal{P} \mid (p, N_1N_2) = 1 \text{ and } \lambda_f(p) \neq \lambda_g(p)\}\,.$$

Now assume that at least one of f or g is a non-CM form and $f \neq g \otimes \chi$ for any Dirichlet character χ. Then using Theorem 3.3.4, we conclude that the set $\tilde{S}$ has natural density one and hence $d(S) = 1$ as $S \setminus \tilde{S}$ is finite.

Next we assume that at least one of f or g is not of CM type and $f = g \otimes \chi$ for some Dirichlet character χ. Note that χ is a non-trivial quadratic character as $f \neq g$ and $a_f(n)$ and $a_g(n)$'s are real for all n. In this case, $\tilde{S}$ is equal to

$$\{p \in \mathcal{P} \mid (p, N_1N_2) = 1 \text{ and } \chi(p) \neq 1\}\,.$$

Since the set

$$\{p \in \mathcal{P} \mid \chi(p) = -1\}$$

has natural density 1/2, the set $\tilde{S}$ and hence S has natural density $d(S) = 1/2$.

Finally, we assume that both f and g are of CM type. If $k_1 \neq k_2$, then we use Theorem 3.4.5 to conclude that the lower natural density of the set $\tilde{S}$ and hence that of S is greater than or equal to 1/2 as $f \neq g$. Now suppose that $k_1 = k_2$. In this set-up, if f and g have CM by different fields, then we use Lemma 3.4.7 to get the conclusion of Theorem 3.5.2. When f and g have CM by the same field, then we use Proposition 3.5.7 to conclude our result. This completes the proof of Theorem 3.5.2.

3.5.4 Proof of Theorem 3.5.6

Recall that $\lambda_f(p) = 2\cos\theta_f(p)$, where $\theta_f(p) \in [0, \pi]$. Then for integers $j \geq 1, m \geq 1$ and primes p with $(p, N) = 1$, using the equation (2.1.6), one can write

$$\lambda_f(p^{jm}) = \begin{cases} jm + 1 & \text{if } \theta_f(p) = 0; \\ (-1)^{jm}(jm+1) & \text{if } \theta_f(p) = \pi; \\ \frac{\sin[(jm+1)\theta_f(p)]}{\sin\theta_f(p)} & \text{otherwise.} \end{cases} \tag{3.5.7}$$

When $j = 1$, the above relation (3.5.7) implies that $\{\lambda_f(p^m)\}_{m\in\mathbb{N}}$ changes sign infinitely often if and only if $\lambda_f(p) \neq 2$ for all primes p with $(p, N) = 1$.

From now on, we shall assume that $j > 1$. By Theorem 2.1.13, one knows that for almost all primes p, if $\lambda_f(p) \notin \{2, 0, -2\}$, then $\theta_f(p)/\pi \notin \mathbb{Q}$. The phenomena $\lambda_f(p) \notin \{2, 0, -2\}$ implies that $\theta_f(p)/\pi \notin \mathbb{Q}$ is true for all primes p with $(p, N) = 1$ when $k \geq 4$.

Now if $\lambda_f(p) \notin \{2, 0, -2\}$, then by applying Weyl's criterion for uniform distribution, we conclude that the sequence $\{(jm+1)\theta_f(p)\}_{m\in\mathbb{N}}$ is uniformly distributed in $[0, 2\pi]$. Thus, in this case, $\{\lambda_f(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often for all j.

If $\lambda_f(p) = 2$, that is, if $\theta_f(p) = 0$, then using (3.5.7), we see that the set

$$\left\{m \in \mathbb{N} \mid \lambda_f(p^{jm}) < 0\right\} = \emptyset$$

for all j.

If $\lambda_f(p) = -2$, that is, if $\theta_f(p) = \pi$, then again using (3.5.7), we have

$$\{m \in \mathbb{N} \mid \lambda_f(p^{jm}) < 0\} \;=\; \emptyset$$

or the sequence $\{\lambda_f(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often depending on j is even or j is odd.

Next assume that $\lambda_f(p) = 0$, that is, $\theta_f(p) = \pi/2$. Then the sequence $\{\lambda_f(p^{jm})\}_m$ changes sign infinitely often if and only if $4 \nmid j$. This completes the proof of Theorem 3.5.6.

3.5.5 Proof of Theorem 3.5.5

Note that sign changes of the sequence $\{a_f(p^{jm})\}_{m\in\mathbb{N}}$ is equivalent to the sign changes of the sequence $\{\lambda_f(p^{jm})\}_{m\in\mathbb{N}}$. Now by Theorem 3.5.6, there exists a natural number M, such that S_j contains a subset $\tilde{S}_j$ defined by

$$\tilde{S}_j \;:=\; \begin{cases} \{p \in \mathcal{P} \mid (p, M) = 1, \lambda_f(p) \neq 2\} & \text{if } j \text{ is odd;} \\ \{p \in \mathcal{P} \mid (p, M) = 1, \lambda_f(p) \notin \{2, -2\}\} & \text{if } j \equiv 2 \mod 4; \\ \{p \in \mathcal{P} \mid (p, M) = 1, \lambda_f(p) \notin \{2, 0, -2\}\} & \text{if } j \equiv 0 \mod 4. \end{cases}$$

Further when $k \geq 4$ or $j = 1$, one can take $M = N$. Since $S_j \setminus \tilde{S}_j$ is a finite set, the natural density of $S_j \setminus \tilde{S}_j$ is zero.

Now if f is of non-CM type, then by the Sato-Tate conjecture which is now a theorem (see Theorem 3.3.3), we see that the set $\tilde{S}_j$ has natural density one.

Next assume that f is a CM form. Then by Theorem 3.4.4, we have the natural density of the set $\tilde{S}_j$ is one if $4 \nmid j$ and the natural density of the set $\tilde{S}_j$ is $1/2$ if $4 \mid j$.

This along with the fact that $d(S_j \setminus \tilde{S}_j) = 0$ implies that S_j has the desired natural densities depending on j.

3.5.6 Proof of Theorem 3.5.3

To prove Theorem 3.5.3, we shall follow the line of action of the proof of Theorem 3.5.1. As in the proof of Theorem 3.5.1, for any prime p with $(p,\ N_1N_2) = 1$, one can write

$$\lambda_f(p) \;:=\; 2\cos\theta_f(p) \quad \text{and} \quad \lambda_g(p) \;:=\; 2\cos\theta_g(p)$$

with $\theta_f(p), \theta_g(p) \in [0, \pi]$. Then for any natural numbers $m, j \in \mathbb{N}$, we have

$$\lambda_f(p^{jm}) \;=\; \begin{cases} jm+1 & \text{if } \theta_f(p) = 0; \\ (-1)^{jm}(jm+1) & \text{if } \theta_f(p) = \pi; \\ \frac{\sin[(jm+1)\theta_f(p)]}{\sin\theta_f(p)} & \text{otherwise;} \end{cases}$$

and

$$\lambda_g(p^{jm}) \;=\; \begin{cases} jm+1 & \text{if } \ \theta_g(p) = 0; \\ (-1)^{jm}(jm+1) & \text{if } \ \theta_g(p) = \pi; \\ \frac{\sin[(jm+1)\theta_g(p)]}{\sin\theta_g(p)} & \text{otherwise.} \end{cases}$$

Note that if $\lambda_f(p) = \lambda_g(p)$, that is, if $\theta_f(p) = \theta_g(p)$, then $\lambda_f(p^{jm}) = \lambda_g(p^{jm})$ for all $m, j \in \mathbb{N}$. Thus for any $j \in \mathbb{N}$, the sequence $\{\lambda_f(p^{jm})\lambda_g(p^{jm})\}_{m\in\mathbb{N}}$ does not change sign.

Now assume that $\lambda_f(p) \neq \lambda_g(p)$, that is, $\theta_f(p) \neq \theta_g(p)$. Further if $\theta_f(p), \theta_g(p) \in \{0, \pi\}$, then we have

$$\lambda_f(p^{jm})\lambda_g(p^{jm}) \;=\; (-1)^{jm}(jm+1)^2$$

for all $m, j \in \mathbb{N}$ and hence the sequence $\{\lambda_f(p^{jm})\lambda_g(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often if and only if j is odd. Henceforth, without loss of generality, we can assume that at least one of $\theta_f(p)$ or $\theta_g(p)$, say $\theta_g(p)$ lies inside the interval $(0, \pi)$. Now for any $m, j \in \mathbb{N}$, we have

$$\lambda_f(p^{jm})\lambda_g(p^{jm}) = \begin{cases} \frac{(jm+1)\sin[(jm+1)\theta_g(p)]}{\sin\theta_g(p)} & \text{if } \theta_f(p) = 0 \text{ and } \theta_g(p) \in (0, \pi); \\ \frac{(jm+1)\sin[(jm+1)(\pi-\theta_g(p))]}{\sin[\pi-\theta_g(p)]} & \text{if } \theta_f(p) = \pi \text{ and } \theta_g(p) \in (0, \pi). \end{cases}$$

Note that both $\theta_g(p)$ and $\pi - \theta_g(p)$ lie inside $(0, \pi)$. Arguing as in Theorem 3.5.6, there exists a natural number M such that for any prime p with $(p, M) = 1$, the sequence $\{\lambda_f(p^{jm})\lambda_g(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often if and only if $4 \nmid j$ or $\theta_g(p) \neq \pi/2$.

Next assume that both $\theta_f(p)$ and $\theta_g(p)$ lie inside $(0, \pi)$. Then we can write

$$\frac{\sin[(jm+1)\theta_f(p)]\sin[(jm+1)\theta_g(p)]}{\sin\theta_f(p)\sin\theta_g(p)} = \frac{\cos[(jm+1)u] - \cos[(jm+1)v]}{2\sin\theta_f(p)\sin\theta_g(p)},$$

where $u := \theta_f(p) - \theta_g(p)$, $v := \theta_f(p) + \theta_g(p)$ and

$$u \not\equiv 0 \mod 2\pi, \quad v \not\equiv 0 \mod 2\pi, \quad u \not\equiv v \mod 2\pi, \quad u \not\equiv -v \mod 2\pi.$$

We claim that $\{\cos[(jm+1)u] - \cos[(jm+1)v]\}_{m\in\mathbb{N}}$ changes sign infinitely often if

$$u \neq \frac{n\pi}{j} \quad \text{and} \quad v \neq \frac{m\pi}{j} \tag{3.5.8}$$

for any integers n, m with $-j < n < j$, $n \neq 0$ and $1 \leq m < 2j$. Suppose not. Then there exists $m_0 \in \mathbb{N}$ such that

$$\big(\cos[(jm+1)u] - \cos[(jm+1)v]\big) \geq 0 \tag{3.5.9}$$

for all $m \geq m_0$. The case $(\cos[(jm+1)u] - \cos[(jm+1)v]) \leq 0$ for all $m \geq m_0$ can be treated similarly. Using (3.5.9), for any $m' \geq m_0$, we have

(3.5.10)
$$\sum_{m=m_0}^{m'} \big(\cos[(jm+1)u] - \cos[(jm+1)v]\big)^2 \;\leq\; 2 \sum_{m=m_0}^{m'} \big(\cos[(jm+1)u] - \cos[(jm+1)v]\big).$$

For any $j \in \mathbb{N}$ and $x \in \mathbb{R}$ with $jx \not\equiv 0 \mod 2\pi$, one knows that

$$\cos(jx) + \cos(2jx) + \cdots + \cos(jnx) \;=\; \cos(jnx/2)\frac{\sin[(n+1)jx/2]}{\sin(jx/2)} - 1 \tag{3.5.11}$$

and
$$\sin(jx) \;+\; \sin(2jx) \;+\; \cdots \;+\; \sin(jnx) = \sin(jnx/2)\frac{\sin[(n+1)jx/2]}{\sin(jx/2)}.$$

Using (3.5.11), one can easily deduce that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \cos[(jm+1)x] \;=\; 0 \tag{3.5.12}$$

provided $jx \not\equiv 0 \mod 2\pi$. Thus if $ju \not\equiv 0 \mod 2\pi$ and $jv \not\equiv 0 \mod 2\pi$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos[(jm+1)u] \;-\; \cos[(jm+1)v]\big) \;=\; 0. \tag{3.5.13}$$

On the other hand, one has

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos[(jm+1)u] \;-\; \cos[(jm+1)v]\big)^2 \\
=\;& 1 \;+\; \lim_{N\to\infty} \frac{1}{2N} \sum_{m=1}^{N} \big(\cos[(jm+1)2u] \;+\; \cos[(jm+1)2v]\big) \\
-\;& \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos[(jm+1)(u+v)] \;+\; \cos[(jm+1)(v-u)]\big).
\end{aligned}$$

Note that $u+v = 2\theta_f(p)$ and $v-u = 2\theta_g(p)$ and by Theorem 2.1.13, for all sufficiently large $p \in \mathcal{P}$, either $\theta_f(p)/\pi$ or $\theta_g(p)/\pi$ are not in $\mathbb{Q}$ except when $\theta_f(p) = \pi/2$ or $\theta_g(p) = \pi/2$. Now using the assumption (3.5.8) and the identity (3.5.12), we see that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} \big(\cos[(jm+1)u] - \cos[(jm+1)v]\big)^2 \;\geq\; 1.$$

This together with (3.5.13) contradicts (3.5.10) and hence proves our claim. In other words, the sequence $\{\lambda_f(p^{jm})\lambda_g(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often when both $\theta_f(p)$ and $\theta_g(p)$ lie inside $(0,\pi)$ and satisfy the assumption (3.5.8).

Recall that both f and g are not of CM type. First suppose that $f \neq g\otimes\chi$ for any Dirichlet character χ. For any $j \in \mathbb{N}$, consider the set $\tilde{S}_j$ defined by

$$S_j' \;:=\; \{p \in \mathcal{P} \mid (p,M)=1, \lambda_f(p) \neq \lambda_g(p) \;\text{ and } u,v \text{ satisfy condition (3.5.8)}\},$$

where M is defined as before. We will show that a subset $\tilde{S}_j$ of $S_j' \cap S_j$ has natural density one and hence $d(S_j)=1$. Set

$$\tilde{S}_j \;:=\; \begin{cases} S_j' & \text{if } j \text{ is odd;} \\ \big\{p \in S_j' \mid \text{ at least one of } \theta_f(p), \theta_g(p) \in (0,\pi)\big\} & \text{if } \; j \equiv 2 \mod 4; \\ \big\{p \in S_j' \mid \text{one of } \theta_f(p), \theta_g(p) \in (0,\pi) \setminus \big\{\frac{\pi}{2}\big\} & \\ \qquad\qquad \text{or both } \theta_f(p), \theta_g(p) \in (0,\pi)\big\} & \text{if } \; j \equiv 0 \mod 4. \end{cases}$$

We have already shown that if $p \in \tilde{S}_j$, then the sequence $\{\lambda_f(p^{jm})\lambda_g(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often. Now using Theorem 3.3.6 and Theorem 3.3.10, we see that the set $\mathcal{P} \setminus S_j'$ has natural density zero and hence $d(S_j') = 1$. Since by the Sato-Tate Theorem 3.3.3, $d(\tilde{S}_j \setminus S_j') = 0$, we have $d(\tilde{S}_j) = 1$ and hence $d(S_j) = 1$.

Now if $f = g \otimes \chi$ for some Dirichlet character χ, then by the given assumptions, it follows that χ is a non-trivial quadratic character. Also for any $j \in \mathbb{N}$ and $p \in \mathcal{P}$, one has

$$\lambda_f(p^{jm}) \;=\; \chi(p^{jm})\lambda_g(p^{jm}) \;=\; \chi(p)^{jm}\lambda_g(p^{jm}).$$

Thus the sequence $\{\lambda_f(p^{jm})\lambda_g(p^{jm})\}_{m\in\mathbb{N}}$ changes sign infinitely often if and only if

j is odd. When j is odd, then $\lambda_f(p^{jm})\lambda_g(p^{jm})$ changes sign if and only if $\chi(p) = -1$. Thus, in this case, the set of primes for which $\lambda_f(p^{jm})\lambda_g(p^{jm})$ changes sign has natural density 1/2. This completes the proof of Theorem 3.5.3.

Remark 3.5.8. *We note that arguing as in Theorem 3.5.1 and Theorem 3.5.3, one can also show that;*

Theorem 3.5.9. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized Hecke eigenforms and p be a prime such that $(p, N_1N_2) = 1$. Then the following conditions are equivalent;*

1. *the sequence $\{\lambda_f(p^{2m})\lambda_g(p^{2m})\}_{m\in\mathbb{N}}$ changes sign infinitely often.*
2. *one has $\lambda_f(p) \neq \pm\lambda_g(p)$.*

Chapter 4

Non-vanishing of Hecke eigenvalues of modular forms

The main aim of this chapter is to study simultaneous non-vanishing of Hecke eigenvalues of elliptic modular forms. Here we consider modular forms of level one as well as higher level. Recall that for integers $k \geq 2,\ N \geq 1$, the space of cusp forms of weight k for the congruence subgroup $\Gamma_0(N)$ is denoted by $S_k(N)$ and the subspace of newforms is denoted by $S_k^{new}(N)$. Let

(4.0.1)
$$f(z) \ := \ \sum_{n=1}^{\infty} a_f(n) q^n \in S_{k_1}^{new}(N_1) \quad \text{and} \quad g(z) \ := \ \sum_{n=1}^{\infty} a_g(n) q^n \in S_{k_2}^{new}(N_2)$$

be distinct normalized Hecke eigenforms. As mentioned earlier, in this case, the normalization ensures that n-th Hecke eigenvalue $\mu_f(n) = a_f(n)$ and $\mu_g(n) = a_g(n)$ for $(n, N_1N_2) = 1$. In this chapter, we study non-vanishing of the sequence $\{a_f(n)a_g(n)\}_{n\in\mathbb{N}}$.

The next section is devoted to the study of first non-vanishing of the above sequence. In the penultimate section, we shall study simultaneous non-vanishing and derive some quantitative results. In the last section, we shall use the properties of

$\mathfrak{B}$-free numbers to investigate non-vanishing of Hecke eigenvalues in short intervals. This chapter is based on a joint work with Gun and Kumar [28].

4.1 First simultaneous non-vanishing

In this section, we shall prove the following theorem.

Theorem 4.1.1. *Let f and g be as in* (4.0.1). *Also let $N := lcm\,[N_1, N_2] > 12$. Then there exists a positive integer $1 < n \leq (2\log N)^4$ with $(n, N) = 1$ such that*

$$a_f(n)a_g(n) \neq 0.$$

Moreover, when N is odd, then there exists an integer $1 < n \leq 16$ with $(n, N) = 1$ such that

$$a_f(n)a_g(n) \neq 0.$$

Note that $a_f(1)a_g(1) = 1$ but we are trying to find the first natural number $n > 1$ with $(n, N) = 1$ for which $a_f(n)a_g(n) \neq 0$. We shall call this the first non-trivial simultaneous non-vanishing.

4.1.1 An intermediate lemma

Proposition 4.1.2. *Let f and g be as in* (4.0.1). *Then for any prime p with $(p, N_1N_2) = 1$, there exists an integer m with $1 \leq m \leq 4$ such that $a_f(p^m)a_g(p^m) \neq 0$.*

Proof of Proposition 4.1.2. Recall from section 2.1.4 that $a_f(p) = p^{\frac{(k_1-1)}{2}} \cos\theta_f(p)$ and $a_g(p) = p^{(k_2-1)/2}\cos\theta_g(p)$ with $0 \leq \theta_f(p), \theta_g(p) \leq \pi$ for any prime $(p, N_1N_2) = 1$. Hence for any $m \in \mathbb{N}$ using the identity (2.1.6) we see that $a_f(p^m)a_g(p^m) \neq 0$ is equivalent to $\sin[(m+1)\theta_f(p)]\sin[(m+1)\theta_g(p)] \neq 0$. Now if $a_f(p)a_g(p) \neq 0$, then we are done. Thus we suppose that $a_f(p)a_g(p) = 0$, then either $a_f(p) = 0$ or $a_g(p) = 0$.

Case (1): If $a_f(p) = 0 = a_g(p)$, then $\theta_f(p) = \theta_g(p) = \pi/2$. Hence we have

$$a_f(p^2)a_g(p^2) \;=\; p^{k_1+k_2-2} \;\neq\; 0.$$

Case (2): Suppose that at least one of $a_f(p)$ or $a_g(p)$ is not zero. Without loss of generality assume that $a_f(p) = 0$ and $a_g(p) \neq 0$, that is, $\theta_f(p) = \pi/2$ and $\theta_g(p) \neq \pi/2$. Now if $\theta_g(p) = 0$ or π, then $a_g(p^2) = 3p^{k_2-1}$. Hence we have

$$a_f(p^2)a_g(p^2) \;=\; -3p^{k_1+k_2-2} \;\neq\; 0.$$

If $\theta_g(p) \notin \{0, \pi/2, \pi\}$, then this implies that $a_g(p^2) = p^{(k_2-1)} \sin[3\theta_g(p)]/\sin\theta_g(p)$. Now if $a_f(p^2)a_g(p^2) = 0$, then $\theta_g(p) \in \{\pi/3, 2\pi/3\}$ as $0 < \theta_g(p) < \pi$. Then we have

$$\frac{a_f(p^4)a_g(p^4)}{p^{2(k_1+k_2-2)}} \;=\; \frac{2}{\sqrt{3}} \sin\frac{5\pi}{2} \sin\frac{5\pi}{3} \quad \text{or} \quad \frac{a_f(p^4)a_g(p^4)}{p^{2(k_1+k_2-2)}} \;=\; \frac{2}{\sqrt{3}} \sin\frac{5\pi}{2} \sin\frac{10\pi}{3}.$$

Since neither $\sin(5\pi/2)\sin(5\pi/3)$ nor $\sin(5\pi/2)\sin(10\pi/3)$ is equal to zero, this completes the proof of Proposition 4.1.2. $\square$

4.1.2 Proof of Theorem 4.1.1

To complete the proof of the first part of Theorem 4.1.1, we first show the existence of a prime $p \leq 2\log N$ with $(p, N) = 1$. By a theorem of Rosser and Schoenfeld (see [84, p. 70]), we know that

$$\sum_{p \leq x} \log p \;>\; 0.73x \quad \text{for all } x \;\geq\; 41.$$

Using this, one can easily check that

$$\sum_{p \leq x} \log p \;>\; \frac{x}{2} \quad \text{for all } x \;\geq\; 5.$$

Now consider the following product

$$\prod_{p \le 2\log N} p \;=\; \exp\left(\sum_{p \le 2\log N} \log p \right) \;>\; N,$$

which confirms the existence of a prime $p \le 2\log N$ such that $(p, N) = 1$. Now we apply Proposition 4.1.2 to complete the proof of the first part of Theorem 4.1.1. The second part of Theorem 4.1.1 follows immediately from Proposition 4.1.2 and the fact that $2 \nmid N$.

4.2 Some quantitative results

In this section, we shall state and prove some quantitative non-vanishing results about the sequence $\{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ and our first theorem in this direction is stated below.

Theorem 4.2.1. *Let f and g be as in* (4.0.1). *Then for any prime* $(p, N_1N_2) = 1$, *the set*

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0\} \tag{4.2.1}$$

has positive density.

Note that Theorem 3 of Gun, Kohnen and Rath [27] showed that for infinitely many primes p, the sequence $\mathcal{A}_p := \{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ has infinitely many sign changes and hence in particular, $\mathcal{A}_p$ has infinitely many non-zero elements. But Theorem 4.2.1 shows that for all primes p with $(p, N_1N_2) = 1$, the non-zero elements of the sequence $\mathcal{A}_p$ has positive density and hence Theorem 4.2.1 does not follow from Theorem 3 of [27]. Our next theorem strengthens Theorem 1.2 of Kumari and R. Murty [45].

Theorem 4.2.2. *Let f and g be as in* (4.0.1). *Further assume that f and g are non-CM forms. Then there exists a set S of primes with natural density one such that for any $p \in S$ and integers $m, m' \geq 1$, we have*

$$a_f(p^m)a_g(p^{m'}) \neq 0.$$

4.2.1 Proof of Theorem 4.2.1

We start by recalling (see Section 2.1.4 for details) that

$$\lambda_f(p) := \frac{a_f(p)}{p^{(k_1-1)/2}} \quad \text{and} \quad \lambda_g(p) := \frac{a_g(p)}{p^{(k_1-1)/2}}$$

and for any prime $(p, N_1N_2) = 1$, we have

$$\lambda_f(p) = 2\cos\theta_f(p) \quad \text{and} \quad \lambda_g(p) = 2\cos\theta_g(p), \quad \text{where} \quad 0 \leq \theta_f(p), \theta_g(p) \leq \pi.$$

Further, for any prime $(p, N_1N_2) = 1$, we have

$$\lambda_f(p^m) = \begin{cases} (-1)^m(m+1) & \text{if } \theta_f(p) = \pi; \\ m+1 & \text{if } \theta_f(p) = 0; \\ \frac{\sin[(m+1)\theta_f(p)]}{\sin\theta_f(p)} & \text{otherwise} \end{cases} \tag{4.2.2}$$

and

$$\lambda_g(p^m) = \begin{cases} (-1)^m(m+1) & \text{if } \theta_g(p) = \pi; \\ m+1 & \text{if } \theta_g(p) = 0; \\ \frac{\sin[(m+1)\theta_g(p)]}{\sin\theta_g(p)} & \text{otherwise.} \end{cases}$$

Now Theorem 4.2.1 follows from the following four cases.

Case (1): When $\theta_f(p) = 0$ or π and $\theta_g(p) = 0$ or π, then by (4.2.2), we see that

$$\{m \in \mathbb{N} \mid a_f(p^m)a_g(p^m) \neq 0\} \;=\; \mathbb{N}.$$

In this case all elements of the sequence $\{a_f(p^m)a_g(p^m)\}_{m\in\mathbb{N}}$ are non-zero.

Case (2): Suppose that at least one of $\theta_f(p), \theta_g(p)$, say $\theta_f(p) = 0$ or π and $\theta_g(p) \in (0, \pi)$. If $\theta_g(p)/\pi \notin \mathbb{Q}$, there is nothing to prove. Now if $\theta_g(p)/\pi = r/s$ with $(r, s) = 1$, then we have

$$\#\{m \le x \mid a_f(p^m)a_g(p^m) \neq 0\} \;=\; \#\{m \le x \mid a_g(p^m) \neq 0\} \;=\; [x] - \left[\frac{x+1}{s}\right].$$

Hence the set $\{m \mid a_f(p^m)a_g(p^m) \neq 0\}$ has postive density.

Case (3): Suppose that $\theta_f(p) = \theta_g(p) \in (0, \pi)$, that is, $\theta_f(p)/\pi = \theta_g(p)/\pi \in (0, 1)$. Now if $\theta_f(p)/\pi \notin \mathbb{Q}$, then $a_f(p^m)a_g(p^m) \neq 0$ for all $m \in \mathbb{N}$ as $\sin m\theta_f(p) \neq 0$ for all $m \in \mathbb{N}$. If $\theta_f(p)/\pi \in \mathbb{Q}$, say $\theta_f(p)/\pi = r/s$, where $r, s \in \mathbb{N}$ with $(r, s) = 1$, then we have $\sin m\theta_f(p) = 0$ if and only if m is an integer multiple of s and hence

$$\#\{m \le x \mid a_f(p^m)a_g(p^m) \neq 0\} \;=\; [x] - \left[\frac{x+1}{s}\right].$$

Hence the set in (4.2.1) has positive density.

Case (4): Assume that $\theta_f(p), \theta_g(p) \in (0, \pi)$ with $\theta_f(p) \neq \theta_g(p)$. If both $\theta_f(p)/\pi$ and $\theta_g(p)/\pi$ are not rational, then there is nothing to prove. Next suppose that one of them, say $\theta_f(p)/\pi$ is rational with $\theta_f(p)/\pi = r/s$ where $(r, s) = 1$ and $\theta_g(p)/\pi \notin \mathbb{Q}$. Then we have

$$\#\{m \le x \mid a_f(p^m)a_g(p^m) \neq 0\} \;=\; \#\{m \le x \mid a_f(p^m) \neq 0\} \;=\; [x] - \left[\frac{x+1}{s}\right].$$

Hence the set in (4.2.1) has positive density.

Now assume both $\theta_f(p)/\pi, \theta_g(p)/\pi \in \mathbb{Q}$. If $\theta_f(p)/\pi = r_1/s_1$ and $\theta_g(p)/\pi = r_2/s_2$ with $(r_i, s_i) = 1$, for $1 \leq i \leq 2$, then

$$\#\{m \leq x \mid a_f(p^m)a_g(p^m) \neq 0\} = \#\left[\{m \leq x | a_f(p^m) \neq 0\} \cap \{m \leq x \mid a_g(p^m) \neq 0\}\right].$$

Since $0 < |r_i/s_i| < 1$ for $i = 1, 2$ and $\theta_f(p) \neq \theta_g(p)$, hence both s_1, s_2 can not be 2. Otherwise, we have $r_1 = r_2 = 1$ and hence $\theta_f(p) = \theta_g(p)$. Now note that

$$\begin{aligned}\#\{m \leq x \mid a_f(p^m)a_g(p^m) = 0\} &= \#\left[\{m \leq x \mid a_f(p^m) = 0\} \cup \{m \leq x \mid a_g(p^m) = 0\}\right] \\ &\leq \left[\frac{x+1}{s_1}\right] + \left[\frac{x+1}{s_2}\right].\end{aligned}$$

Hence the set in (4.2.1) has positive density. This completes the proof of Theorem 4.2.1.

4.2.2 Proof of Theorem 4.2.2

For any real $x \geq 2$ and $0 < \delta < 1/2$, using Theorem 2.1.16, we have

$$\#\{p \leq x \mid a_f(p^m) = 0 \text{ for some } m \geq 1\} \ll_{f,\delta} \frac{x}{(\log x)^{1+\delta}},$$

where the implied constant depends only on f and δ. We have the same estimate for the eigenform g as well. Therefore for any $x \geq 2$ and $0 < \delta < 1/2$, we have

$$\#\left\{p \leq x \mid a_f(p^m)a_g(p^{m'}) = 0 \text{ for some } m, m' \geq 1\right\} \ll_{f,g,\delta} \frac{x}{(\log x)^{1+\delta}}, \tag{4.2.3}$$

where the implied constant depends on f, g and δ. Hence

$$\begin{aligned}&\#\left\{p \leq x \mid a_f(p^m)a_g(p^{m'}) \neq 0 \text{ for all } m, m' \geq 1\right\} \\ &= \pi(x) - \#\left\{p \leq x \mid a_f(p^m)a_g(p^{m'}) = 0 \text{ for some } m, m' \geq 1\right\},\end{aligned}$$

where $\pi(x)$ denotes the number of primes up to x. Now using prime number theorem as well as the identity (4.2.3), we have

$$\#\left\{p \leq x \mid a_f(p^m)a_g(p^{m'}) \neq 0 \text{ for all } m, m' \geq 1\right\} \sim \frac{x}{\log x}.$$

Hence the set

$$\left\{p \in \mathcal{P} \mid a_f(p^m)a_g(p^{m'}) \neq 0 \text{ for any integers } m, m' \geq 1\right\}$$

has natural density one.

4.3 Simultaneous non-vanishing and $\mathfrak{B}$-free numbers

In this section, we shall use known properties of $\mathfrak{B}$-free numbers to study simultaneous non-vanishing of Hecke eigenvalues of two distinct Hecke eigenforms which lie in the newform space. Motivated by the study of non-vanishing of the Ramanujan τ-function, Serre [91, page 383] initiated the study of estimating the size of possible gaps in Hecke eigenvalues (more generally, Fourier coefficients). In this connection, for any $f \in S_k(N)$, he defined the gap function as follows:

$$i_f(n) := \max \left\{m \in \mathbb{N} \;\middle|\; a_f(n+j) = 0 \text{ for all } 0 < j \leq m\right\}.$$

He proved that $i_f(n) = O(n)$ if $f \in S_k(N)$ is a non-CM form and asked for an estimate of type $i_f(n) = O(n^\theta)$ for some $\theta < 1$. A stronger form of this question is to find the smallest $\theta < 1$ such that

$$\#\left\{x < n < x + x^\theta \;\middle|\; a_f(n) \neq 0\right\} \gg x^\theta.$$

These questions have venerable history. Right after the result of Serre, K. Murty [63] showed that $i_f(n) = O(n^{3/5})$. Balog and Ono [8] were the first to use properties of $\mathfrak{B}$-free numbers to study these questions. Alkan and Zaharescu [2] proved that

$$i_\Delta(n) \ll_\Delta n^{1/4+\epsilon}$$

for the Ramanujan Δ-function. Kowalski, Robert and Wu [43], using distribution of $\mathfrak{B}$-free numbers in short intervals showed that

$$i_f(n) \ll_f n^{7/17+\epsilon}$$

where $f \in S_k^{new}(N)$ is a non-CM normalized Hecke eigenform. Recently, Das and Ganguly [15] showed that

$$i_f(n) \ll_f n^{1/4+\epsilon}$$

for any $f \in S_k(1)$.

In this section, for any $f \in S_{k_1}(N_1)$ and $g \in S_{k_2}(N_2)$, we shall consider the question of estimating of the following generalized gap function

$$i_{f,g}(n) := \max \left\{ m \in \mathbb{N} \;\middle|\; a_f(n+j)a_g(n+j) = 0 \text{ for all } 0 < j \leq m \right\}$$

and its stronger form, that is, the question of finding the smallest $\theta < 1$ such that

$$\# \left\{ x < n < x + x^\theta \;\middle|\; a_f(n)a_g(n) \neq 0 \right\} \gg x^\theta .$$

These questions were first considered by Kumari and R. Murty [45]. Here we first consider a generalization of the above question. More precisely, we show the following.

Theorem 4.3.1. *Let* $h : \mathbb{N} \to \mathbb{C}$ *be a multiplicative function and* $N \geq 1$ *be an* *integer. Define*

$$(4.3.1) \qquad \mathfrak{P}_{h,N} \; := \; \{p \in \mathcal{P} \mid h(p) = 0\} \; \cup \; \{p \in \mathcal{P} \mid p \mid N\}.$$

Also assume that $\mathfrak{P}_{h,N}$ *satisfies condition* (2.4.1). *Then*

1. *for any* $\epsilon > 0$ *there exists* $x_0(\mathfrak{P}_{h,N},\ \epsilon) > 0$ *such that for all* $x \geq x_0(\mathfrak{P}_{h,N},\ \epsilon)$ *and* $y \geq x^{\theta(\rho)+\epsilon}$, *we have*

$$\#\left\{x < n \leq x+y \;\middle|\; (n,N) = 1,\ n \text{ square-free and } h(n) \neq 0\right\} \gg_{\mathfrak{P}_{h,N},\ \epsilon} y,$$

   where $\theta(\rho)$ *is as in* (2.4.3).

2. *for any* $\epsilon > 0$ *there exists* $x_0(\mathfrak{P}_{h,N},\ \epsilon) > 0$ *such that for all* $x \geq x_0(\mathfrak{P}_{h,N},\ \epsilon)$, $y \geq x^{\psi(\rho)+\epsilon}$ *and* $1 \leq a \leq q \leq x^{\epsilon}$ *with* $(a,q) = 1$, *we have*

$$\#\{x < n \leq x+y \mid (n,N) = 1,\ n \text{ square-free, } n \equiv a (\text{mod } q) \text{ and } h(n) \neq 0\} \gg_{\mathfrak{P}_{h,N},\ \epsilon} \frac{y}{q},$$

   where $\psi(\rho)$ *is as in* (2.4.4).

Proof of Theorem 4.3.1. Let us define

$$\mathfrak{B}_{\mathfrak{P}_{h,N}} \; := \; \mathfrak{P}_{h,N} \; \cup \; \left\{p^2 \mid p \in \mathcal{P} \setminus \mathfrak{P}_{h,N}\right\}.$$

Then the first part of Theorem 4.3.1 follows from Theorem 2.4.3. Applying Theorem 2.4.4, we get the second part of Theorem 4.3.1. □

As an immediate corollary, we have the following.

Corollary 4.3.2. *Let* $E_1/\mathbb{Q}$ *and* $E_2/\mathbb{Q}$ *be two non-CM elliptic curves which have* *the same conductor N. Let*

$$L(E_i, s) \;=\; \sum_{n=1}^{\infty} a_{E_i}(n) n^{-s}, \quad i = 1, 2$$

be their Hasse-Weil L-functions*. Then*

1. *for any $\epsilon > 0$ there exists $x_0(E_1, E_2, \epsilon) > 0$ such that*

$$\#\left\{x < n < x + y \mid n \textit{ is square-free and } a_{E_1}(n) a_{E_2}(n) \neq 0\right\} \;\gg_{E_1, E_2, \epsilon}\; y$$

   for all $x > x_0(E_1, E_2, \epsilon)$ and $y \geq x^{33/94+\epsilon}$.

2. *for any $\epsilon > 0$ there exists $x_0(E_1, E_2, \epsilon) > 0$ such that for any $x \geq x_0(E_1, E_2, \epsilon)$, $y \;\geq\; x^{87/214+\epsilon}$ and $1 \leq a \leq q \leq x^{\epsilon}$ with $(a, q) = 1$, we have*

$$\#\left\{x < n \leq x + y \mid (n, N) = 1,\ n \textit{ is square-free and } n \equiv a \mod q \textit{ and } a_{E_1}(n) a_{E_2}(n) \neq 0\right\} \;\gg_{E_1, E_2, \epsilon}\; \frac{y}{q}.$$

Proof of Corollary 4.3.2. By a work of Elkies [21] , we have

$$\#\big\{p \leq x \mid a_E(p) = 0\big\} \;\ll_E\; x^{3/4}.$$

Considering $h(n) := a_{E_1}(n) a_{E_2}(n)$, one easily sees that $\mathfrak{P}_{h,N}$ satisfies condition (2.4.1) with $\rho = 3/4$ and $\eta = 0$. We now apply Theorem 4.3.1 to conclude the Corollary. $\square$

Recently, Kumari and R. Murty [45, Theorem 1.3] proved similar results for non-CM normalized Hecke eigenforms which are newforms of weight $k > 2$. But here we prove the theorem for non-CM elliptic curves which is equivalent to proving the same for the non-CM normalized Hecke eigenforms of weight 2 lying in the newform spaces.

Another application of the Theorem 4.3.1, we have the following simultaneous non-vanishing of coefficients of symmetric power L-functions in short intervals. To state the result, we need to introduce few more notations. Let $f \in S_k^{new}(N)$ be a normalized Hecke eigenform with the Fourier coefficients $\{a_f(n)\}_{n\in\mathbb{N}}$. Set $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$ and suppose that for prime $p \nmid N$, the Satake p-parameter of f are $\alpha_{f,p}, \beta_{f,p}$. Then the un-ramified m-th symmetric power L-function of f is defined as follows:

$$L_{unr}(sym^m f, s) \ := \ \prod_{p\nmid N} \prod_{0\le j\le m} (1 - \alpha_{f,p}^{j} \beta_{f,p}^{m-j} p^{-s})^{-1} \ := \ \sum_{n\ge 1} \lambda_f^{(m)}(n) n^{-s}.$$

With these notations in place, we now have the following corollary.

Corollary 4.3.3. *Let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized non-CM Hecke eigenforms. Let $N := lcm[N_1, N_2]$. Then*

1. *for any $\epsilon > 0$ there exists $x_0(f, g, \epsilon) > 0$ such that*

$$\#\Big\{x < n \le x + y \mid (n, N) = 1, \ n \text{ is square-free and } \lambda_f^{(m)}(n)\lambda_g^{(m)}(n) \ne 0\Big\} \gg_{f,g,m,N,\epsilon} y$$

   for all $x \ge x_0(f, g, \epsilon)$ and $y \ge x^{7/17+\epsilon}$.

2. *for any $\epsilon > 0$ there exists $x_0(f, g, \epsilon) > 0$ such that for all $x \ge x_0(f, g, \epsilon)$, $y \ge x^{17/38+\epsilon}$ and $1 \le a \le q \le x^{\epsilon}$ with $(a, q) = 1$, we have*

$$\#\{x < n \le x + y \ \mid \ (n, N) = 1, \ n \text{ square-free, } n \equiv a (mod\ q) \text{ and } \lambda_f^{(m)}(n)\lambda_g^{(m)}(n) \ne 0\} \gg_{f,g,m,N,\epsilon} \frac{y}{q}.$$

Proof of Corollary 4.3.3. Let us assume that

$$\mathfrak{P}_{f,g,m,N} \ := \ \Big\{p \in \mathcal{P} \mid p \mid N \ \text{ or } \ \lambda_f^{(m)}(p)\lambda_g^{(m)}(p) \ = \ 0\Big\}.$$

Since $\lambda_f^{(m)}(p) = \lambda_f(p^m)$, by Theorem 2.1.16 (also see Theorem 2.1.15) we see that $\mathfrak{P}_{f,g,m,N}$ satisfies condition (2.4.1). Note that $h(n) := \lambda_f^{(m)}(n)\lambda_g^{(m)}(n)$ is a multiplicative function and hence we can apply Theorem 4.3.1 to complete the proof of Corollary 4.3.3. $\square$

Remark 4.3.4. *Note that Corollary 4.3.3 implies simultaneous non-vanishing of Hecke eigenvalues in sparse sequences. More precisely, let $f \in S_{k_1}^{new}(N_1)$ and $g \in S_{k_2}^{new}(N_2)$ be normalized non-CM Hecke eigenforms. Also let $N := lcm[N_1, N_2]$. Then*

1. *for any $\epsilon > 0$ there exists $x_0(f, g, \epsilon) > 0$ such that*

$$\#\{x < n \leq x + y \mid (n, N) = 1,\ n \text{ is square-free and } \lambda_f(n^m)\lambda_g(n^m) \neq 0\} \gg_{f,g,m,N,\epsilon} y$$

for all $x \geq x_0(f, g, \epsilon)$ and $y \geq x^{7/17+\epsilon}$.

2. *for any $\epsilon > 0$ there exists $x_0(f, g, \epsilon) > 0$ such that for all $x \geq x_0(f, g, \epsilon)$, $y \geq x^{17/38+\epsilon}$ and $1 \leq a \leq q \leq x^{\epsilon}$ with $(a, q) = 1$, we have*

$$\#\Big\{x < n \leq x + y \ \mid\ (n, N) = 1,\ n \text{ square-free, } n \equiv a (\text{mod } q) \text{ and } \lambda_f(n^m)\lambda_g(n^m) \neq 0\Big\} \gg_{f,g,m,N,\epsilon} \frac{y}{q}.$$

Chapter 5

Hecke eigenvalues of Siegel modular forms of degree two: I

5.1 Introduction

This chapter is devoted to the study of arithmetic properties of Hecke eigenvalues of Hecke eigenforms which lie in the Maass subspace of the space of Siegel cusp forms degree two. Recall that the Maass subspace is defined by

$$S_k^* := \left\{ F \in S_k(\Gamma_2) \;\middle|\; a_F \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} = \sum_{d|(n,m,r)} d^{k-1} a_F \begin{pmatrix} nm/d^2 & r/2d \\ r/2d & 1 \end{pmatrix} \right.$$
$$\left. \forall \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} > 0 \right\},$$

where $a_F(M)$ is the Fourier coefficient of $F \in S_k(\Gamma_2)$. Here, as before, $\Gamma_2 := \mathrm{Sp}_2(\mathbb{Z})$ and $S_k(\Gamma_2)$ denote the Siegel modular group of degree two and the space of cuspidal Siegel modular forms of weight k and degree two respectively. When f is an elliptic cuspform which is a Hecke eigenform of weight k with Hecke eigenvalues $\mu_f(n)$, then by a celebrated result of Deligne [19, Theorem 8.2], we know that for any positive integer $n \in \mathbb{N}$

$$|\mu_f(n)| \ \leq \ d(n) \ n^{(k-1)/2},$$

where $d(n)$ is the number of divisors of n.

Now one would like to know the optimality of the above result, that is, an omega result for the sequence $\{\mu_f(n)/n^{(k-1)/2}\}_{n\in\mathbb{N}}$. This has been researched extensively. In 1973, Rankin [81] showed that

$$\limsup_{n\to\infty} \frac{\mu_f(n)}{n^{(k-1)/2}} \ = \ +\infty.$$

In 1983, R. Murty [64] showed that

$$\mu_f(n) \ = \ \Omega_{\pm}\left(n^{(k-1)/2}\exp\left(\frac{c\log n}{\log\log n}\right)\right),$$

where $c > 0$ is an absolute constant.

In case of Siegel modular forms of degree two, the generalised Ramanujan-Petersson conjecture (see [73]) predicts that

$$\mu_F(p) \ \ll_\epsilon \ p^{k-3/2+\epsilon} \tag{5.1.1}$$

for any prime p and $\epsilon > 0$. It is known that the elements lying the Maass subspace of $S_k(\Gamma_2)$ are the ones which fail to satisfy equation (5.1.1). Thus these Hecke eigenvalues are inaccessible via the Ramanujan-Petersson bounds. Hence we investigate upper and lower bounds, omega result and distributions of these Hecke eigenvalues, that is, the Hecke eigenvalues of an eigenform lying in the Maass subspace.

In this direction, the best known upper bound for these Hecke eigenvalues are due to Pitale and Schmidt [75], which states that for any $\epsilon > 0$, we have

$$\mu_F(n) \ll_\epsilon n^{k-1+\epsilon}.$$

On the other hand, Das and Sengupta [16] proved an omega result for these Hecke eigenvalues. But there is a considerable gap between the known upper bound and the known omega result. In our study, we improve upper bound and omega result for these Hecke eigenvalues. This chapter is based on a joint work with Gun and Sengupta [31].

5.2 Statement of the Theorems

Let S_k^* be the Maass subspace of $S_k(\Gamma_2)$. We first investigate the upper bound for the Hecke eigenvalues $\mu_F(n)$, where $F \in S_k^*$ is a Hecke eigenform.

Theorem 5.2.1. *Let $F \in S_k^*$ be a non-zero Hecke eigenform with Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$. Then there exists an absolute constant $c_1 > 0$ such that*

$$\mu_F(n) \leq n^{k-1} exp\left(c_1\sqrt{\frac{\log n}{\log\log n}}\right)$$

for all integer $n \geq 3$.

Remark 5.2.2. *Theorem 5.2.1 improves an earlier result of Pitale and Schmidt (see page 101 of [75]).*

Our next theorem shows that the above upper bound is not far from being optimal.

Theorem 5.2.3. *Let $F \in S_k^*$ be a non-zero Hecke eigenform with Hecke eigenvalues*

$\{\mu_F(n)\}_{n\in\mathbb{N}}$. *Then there exists an absolute constant* $c > 0$ *such that*

$$\mu_F(n) \;=\; \Omega\left(n^{k-1} exp\left(c\frac{\sqrt{\log n}}{\log\log n}\right)\right).$$

Theorem 5.2.3 strengthens an earlier result of Das and Sengupta [16]. We also have the following lower bound.

Theorem 5.2.4. *Let* $F \in S_k^*$ *be a non-zero Hecke eigenform with Hecke eigenvalues* $\{\mu_F(n)\}_{n\in\mathbb{N}}$. *Then there exist absolute constants* $c_2, c_3 > 0$ *such that*

$$\mu_F(n) \;\geq\; c_2 n^{k-1} exp\left(-c_3\sqrt{\frac{\log n}{\log\log n}}\right)$$

for all integers $n \geq 3$.

As a corollary, we have an alternative proof of the following result of Breulmann [12].

Corollary 5.2.5. *If* $F \in S_k^*$ *is a non-zero Hecke eigenform with Hecke eigenvalues* $\mu_F(n)$, *then* $\mu_F(n) > 0$.

Since $\mu_F(n)/n^{k-1} > 0$, one might wonder whether this result is optimal, that is, ask for a constant $d > 0$ such that $\mu_F(n)/n^{k-1} \geq d$ for all $n \in \mathbb{N}$. Our next theorem precludes such a possibility.

Theorem 5.2.6. *Let* $F \in S_k^*$ *be a non-zero Hecke eigenform with Hecke eigenvalues* $\mu_F(n)$. *Then*

$$\liminf_{n\to\infty} \frac{\mu_F(n)}{n^{k-1}} = 0.$$

Finally, we investigate distributions of limit points of the sequence $\{\mu_F(n)/n^{k-1}\}_n$.

Theorem 5.2.7. *Let* $F \in S_k^*$ *be a non-zero Hecke eigenform with Hecke eigenvalues* $\mu_F(n)$. *Then there are infinitely many limit points of the sequence* $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$ *in* $(1,\infty)$ *and infinitely many of them are in* $(0,1)$.

5.3 Some requisites

We shall need the following lemma to prove Theorem 5.2.3 and Theorem 5.2.6.

Lemma 5.3.1. *Let* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^1$ *be a normalised Hecke eigenform. Then there exist absolute constants* $\beta,\ \beta_1$ *satisfying* $0 < \beta,\ \beta_1 < 2$ *such that both the sets*

$$\left\{p \in \mathcal{P} \mid a(p) < -\beta_1 \cdot p^{(k-1)/2}\right\} \quad \text{and} \quad \left\{p \in \mathcal{P} \mid a(p) > \beta \cdot p^{(k-1)/2}\right\}$$

have positive lower density.

Proof of Lemma 5.3.1. A proof of this lemma can be found in [64, Corollary 2] and in [16, Lemma 3.1]. For the sake of completeness, we give a proof here.

Let $b(p) := a(p)/p^{(k-1)/2}$. Then for any absolute constant $\tilde{\beta}$ satisfying $0 < \tilde{\beta} < 2$, consider the sums

$$S(x) := \sum_{p \le x} \left(b(p) + \tilde{\beta}\right) \left(b(p) - 2\right) \quad \text{and} \quad S^+(x) := \sum_{\substack{p \le x, \\ b(p) < -\tilde{\beta}}} \left(b(p) + \tilde{\beta}\right) \left(b(p) - 2\right).$$

By Deligne's bound $|b(p)| \le 2$, we have

$$S(x) \ \le\ S^+(x) \ \le\ 16\,\#\left\{p \in \mathcal{P} \mid b(p) < -\tilde{\beta}\right\}.$$

Now using the estimates (see pages 43 and 135 of [35] and Theorem 2 of [80])

$$\sum_{p \le x} b(p)\log p \ \ll\ x\,\exp(-\kappa\sqrt{\log x}), \quad \sum_{p \le x} b^2(p)\log p \ \sim\ x \quad \text{and} \quad \sum_{p \le x} 1 \ \sim\ \frac{x}{\log x},$$

where $\kappa > 0$ is an absolute constant, we have

$$\#\left\{p \in \mathcal{P} \mid b(p) < -\tilde{\beta}\right\} \ \ge\ \frac{1}{16} S(x) \ \ge\ \frac{1}{32}\left(1 - 2\tilde{\beta}\right)\frac{x}{\log x} \ +\ o\left(\frac{x}{\log x}\right).$$

Therefore, there exists an absolute constant β_1 satisfying $0 < \beta_1 < 2$ such that

$$\left\{p \in \mathcal{P} \mid a(p) < -\beta_1 \cdot p^{k-3/2}\right\}$$

has positive lower density. Using similar arguments one can show that the other set also has positive lower density. $\square$

One can use the Sato-Tate conjecture (now a theorem due to Barnet-Lamb, Geraghty, Harris and Taylor [10]) to get the above result. But the proof of the lemma avoids this deep theorem.

We recall that by Theorem 2.2.14, if $F \in S_k^*$ is a Hecke eigenform corresponding to the eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$, then there exists a normalized Hecke eigenform $f \in S_{2k-2}(\Gamma_1)$ with the eigenvalues $\{a(n)\}_{n\in\mathbb{N}}$ such that

$$\zeta(2s-2k+4)\sum_{n=1}^{\infty}\frac{\mu_F(n)}{n^s} = \zeta(s-k+1)\zeta(s-k+2)\sum_{n=1}^{\infty}\frac{a(n)}{n^s}. \tag{5.3.1}$$

5.4 Proof of the Theorems

5.4.1 Proof of Theorem 5.2.1

From the identity (5.3.1), for all $m \in \mathbb{N}$ and any $p \in \mathcal{P}$, one can easily deduce that

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} = 1 + \frac{1}{p} + \left(1+\frac{1}{p}\right)\sum_{\ell=1}^{m-1}\frac{a(p^\ell)}{p^{\ell(k-1)}} + \frac{a(p^m)}{p^{m(k-1)}}$$

with the convention that an empty sum is zero. Note that for any $|\lambda| < 1$, we have

$$\sum_{n=2}^{\infty}(n+1)\lambda^n = \sum_{n=3}^{\infty}n\lambda^{n-1} = \frac{3\lambda^2-2\lambda^3}{(1-\lambda)^2}.$$

This can be seen by considering the power series

$$h(x) \;=\; \sum_{n\geq 3} x^n \;=\; \frac{1}{1-x} - 1 - x - x^2$$

and noting that

$$h'(x) \;=\; \frac{3x^2 - 2x^3}{(1-x)^2},$$

where h' is the derivative of h. For any $p \in \mathcal{P}$, let us set

$$\alpha_p \;:=\; \sum_{n=2}^{\infty} \frac{n+1}{p^{n/2}} \;=\; \frac{3p^{1/2} - 2}{p^{1/2}(p^{1/2}-1)^2}.$$

By the work of Deligne [19], one knows that

$$\frac{a(n)}{n^{k-3/2}} \;\leq\; d(n),$$

where $d(n)$ denotes the number of divisors of n. This shows that for any $p \in \mathcal{P}$ and $m \in \mathbb{N}$ with $m \geq 2$, we have

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} \;\leq\; 1 + \frac{1}{p} \;+\; \left(1 + \frac{1}{p}\right)\frac{2}{p^{1/2}} \;+\; \left(1 + \frac{1}{p}\right)\alpha_p.$$

Note that $\alpha_p \;\asymp\; 1/p$. Hence there exists an absolute constant $c_7 > 0$ such that

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} \;\leq\; 1 + \frac{c_7}{p^{1/2}}$$

for all $m \in \mathbb{N}$. Let $n \geq 3$ be an arbitrary natural number and let $t = \nu(n)$ be its number of distinct prime divisors. Then we can write n as

$$n = p_1^{m_1} \cdots p_t^{m_t}$$

where $p_1 < \cdots < p_t$ and $m_i > 0$ for $1 \le i \le t$. Now using the fact that $\log(1+x) \le x$ for any $x > 0$, we have

$$\frac{\mu_F(n)}{n^{k-1}} \le \prod_{1\le i\le t}\left(1+\frac{c_7}{p_i^{1/2}}\right) = \exp\left(\sum_{1\le i\le t}\log\left(1+\frac{c_7}{p_i^{1/2}}\right)\right) \le \exp\left(c_7\sum_{1\le i\le t}\frac{1}{p_i^{1/2}}\right).$$

Since $i < p_i$, we have

$$\frac{\mu_F(n)}{n^{k-1}} \le \exp\left(c_7\sum_{1\le i\le t}\frac{1}{i^{1/2}}\right) \le \exp\left(c_8\, t^{1/2}\right),$$

where $c_8 > 0$ is an absolute constant. Note that $t = \nu(n) \ll \log n/\log\log n$ for $n \gg 1$ (see [98], page 83 for details). Thus for any $n \ge 3$, we have

$$\frac{\mu_F(n)}{n^{k-1}} \le \exp\left(c_1\sqrt{\frac{\log n}{\log\log n}}\right),$$

where $c_1 > 0$ is an absolute constant. This completes the proof of the theorem.

5.4.2 Proof of Theorem 5.2.3

Using the identity (5.3.1), for any prime p, one has

$$\mu_F(p) = p^{k-1}\left(1+\frac{1}{p}+\frac{a(p)}{p^{k-1}}\right).$$

Note that by Lemma 5.3.1, there exists an absolute constant $0 < \beta < 2$ such that the set

$$A := \left\{p \in \mathcal{P} \mid a(p) > \beta \cdot p^{k-3/2}\right\}$$

has positive lower density. For any real $x > 0$, let

$$n_x := \prod_{\substack{5\le p\le x,\\ p\in A}} p$$

with the convention that an empty product is 1. Then for sufficiently large real number $x > 0$, we have

$$\begin{aligned}
\frac{\mu_F(n_x)}{n_x^{k-1}} &= \prod_{\substack{5\le p\le x,\\ p\in A}} \left(1+\frac{1}{p}+\frac{a(p)}{p^{k-1}}\right) \ \ge\ \prod_{\substack{5\le p\le x,\\ p\in A}} \left(1+\frac{a(p)}{p^{k-1}}\right)\\
&\ge \exp\left[\sum_{\substack{5\le p\le x,\\ p\in A}} \log\left(1+\frac{\beta}{p^{1/2}}\right)\right]\\
&\ge \exp\left(c_4 \sum_{\substack{5\le p\le x,\\ p\in A}} \frac{1}{p^{1/2}}\right),
\end{aligned}$$

where $c_4 > 0$ is an absolute constant. Since the set A has positive lower density, using partial summation formula, one can easily see that

$$\sum_{\substack{5\le p\le x,\\ p\in A}} \frac{1}{p^{1/2}} \gg \frac{\sqrt{x}}{\log x},$$

where the implied constant is absolute. Further for any positive real x, we have

$$\log(n_x) = \sum_{\substack{5\le p\le x,\\ p\in A}} \log p \ll x.$$

Note that $\sqrt{x}/\log x$ is an increasing function for $x \ge 8$. Thus for sufficiently large x, we have

$$\frac{\mu_F(n_x)}{n_x^{k-1}} \ge \exp\left(c_5\frac{\sqrt{x}}{\log x}\right) \ge \exp\left(c\frac{\sqrt{\log n_x}}{\log\log n_x}\right),$$

where $c, c_5 > 0$ are absolute constants. This shows that given any natural number M, there exists a natural number n with $n > M$ such that

$$\frac{\mu_F(n)}{n^{k-1}} \ge \exp\left(c\frac{\sqrt{\log n}}{\log\log n}\right).$$

This completes the proof of Theorem 5.2.3.

5.4.3 Proof of Theorem 5.2.4

As earlier by the identity (5.3.1) for any $p \in \mathcal{P}$, we know that

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} \;=\; 1+\frac{1}{p} \;+\; \left(1+\frac{1}{p}\right)\sum_{\ell=1}^{m-1}\frac{a(p^\ell)}{p^{\ell(k-1)}} \;+\; \frac{a(p^m)}{p^{m(k-1)}}.$$

Proceeding as in subsection 5.4.1, for any $p \in \mathcal{P}$ and $m \in \mathbb{N}$ with $m \geq 2$, we see that

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} \;\geq\; 1+\frac{1}{p} \;+\; \left(1+\frac{1}{p}\right)\frac{a(p)}{p^{k-1}} \;-\; \left(1+\frac{1}{p}\right)\alpha_p.$$

Since for any prime $p \geq 11$, one has $\alpha_p < 6/p$ and hence for any integer $m \geq 1$, we have

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} \;\geq\; 1-\frac{1}{p^{1/2}}\left(2+\frac{5}{p^{1/2}}+\frac{2}{p}+\frac{6}{p^{3/2}}\right).$$

Thus except for finitely many primes p, there exists an absolute constant $c_{10} > 0$ such that for all $m \in \mathbb{N}$, we have

$$\text{(5.4.1)} \qquad \frac{\mu_F(p^m)}{p^{m(k-1)}} \;\geq\; 1-\frac{c_{10}}{p^{1/2}} \quad \text{with} \quad \frac{c_{10}}{p^{1/2}} \;<\; 1.$$

It is easy to see that one can choose $c_{10} = 3 \cdot 5$ and hence the inequality (5.4.1) is true for any prime $p \geq$ 17. Let

$$T \;:=\; \{p \in \mathcal{P} \mid \text{ the inequality (5.4.1) holds}\}$$

and $n \in \mathbb{N}$ be any natural number whose prime divisors are in T. As in subsection 5.4.1, writing

$$n = \prod_{1\leq i\leq t} p_i^{m_i}$$

with $m_i > 0$ and $p_1 < \cdots < p_t$, we have

$$\begin{aligned}
\frac{\mu_F(n)}{n^{k-1}} \;\geq\; \prod_{1\leq i\leq t}\left(1-\frac{c_{10}}{p_i^{1/2}}\right) \;&=\; \exp\left(\sum_{1\leq i\leq t}\log\left(1-\frac{c_{10}}{p_i^{1/2}}\right)\right)\\
&\geq\; \exp\left(-c_{11}\sum_{1\leq i\leq t}\frac{1}{p_i^{1/2}}\right)\\
&\geq\; \exp\left(-c_{11}\sum_{1\leq i\leq t}\frac{1}{i^{1/2}}\right) \;\geq\; \exp\left(-c_{12}\, t^{1/2}\right),
\end{aligned}$$

where $c_{11},\ c_{12} > 0$ are absolute constants. Again since $t = \nu(n) \ll \log n/\log\log n$ for $n \gg 1$, hence for such $n \in \mathbb{N}$ with $n \geq 3$, we have

$$\frac{\mu_F(n)}{n^{k-1}} \;\geq\; \exp\left(-c_3\sqrt{\frac{\log n}{\log\log n}}\right), \tag{5.4.2}$$

where $c_3 > 0$ is an absolute constant. Note that (5.4.2) holds if all the prime divisors of n are in the set T. Now if $n \in \mathbb{N}$ is such that $p|n \Rightarrow p \notin T$, then we use Hecke relation

$$a(p^{n+1}) \;=\; a(p)a(p^n) - p^{2k-3}a(p^{n-1})$$

for $n \in \mathbb{N}$ and explicit calculations using Mathematica. In particular, we have

$$\frac{\mu_F(n)}{n^{k-1}} \;\geq\; c_2, \tag{5.4.3}$$

where $c_2 > 0$ is an explicit constant. Combining (5.4.2) and (5.4.3), we now get

$$\mu_F(n) \;\geq\; c_2 n^{k-1}\exp\left(-c_3\sqrt{\frac{\log n}{\log\log n}}\right)$$

for any natural number $n \in \mathbb{N}$ with $n \geq 3$.

5.4.4 Proof of Corollary 5.2.5

Since μ_F is a non-zero multiplicative function (see [3], [71]), we have $\mu_F(1) = 1 > 0$. Further note that

$$\mu_F(2) \;\geq\; \frac{3}{2} \;-\; \sqrt{2} \;>\; 0.$$

Now by applying Theorem 5.2.4, we conclude the corollary.

5.4.5 Proof of Theorem 5.2.6

Again we start by noting that for any prime p, we have

$$\mu_F(p) \;=\; p^{k-1}\left(1 + \frac{1}{p} + \frac{a(p)}{p^{k-1}}\right).$$

By Lemma 5.3.1, there exists an absolute constant $0 < \beta_1 < 2$ such that the set

$$B \;:=\; \left\{p \mid a(p) < -\beta_1 \cdot p^{k-3/2}\right\}$$

has positive lower density. Let us take

$$n_x \;=\; \prod_{\substack{x<p\leq 2x\\ p\in B}} p,$$

where x is sufficiently large so that $2/\sqrt{x} < \beta_1$. Then we have

$$\begin{aligned}
\frac{\mu_F(n_x)}{n_x^{k-1}} \;=\; \prod_{\substack{x<p\leq 2x,\\ p\in B}}\left(1 + \frac{1}{p} + \frac{a(p)}{p^{k-1}}\right) \;&\leq\; \prod_{\substack{x<p\leq 2x,\\ p\in B}}\left(1 + \frac{1}{p} + \frac{-\beta_1}{p^{1/2}}\right)\\
&\leq\; \exp\left[\sum_{\substack{x<p\leq 2x,\\ p\in B}} \log\left(1 - \frac{\beta_1}{2p^{1/2}}\right)\right]\\
&\leq\; \exp\left(-c_{13}\sum_{\substack{x<p\leq 2x,\\ p\in B}} \frac{1}{p^{1/2}}\right),
\end{aligned}$$

where $c_{13} > 0$ is an absolute constant. Since the set B has positive lower density, as in subsection 5.4.2, we get

$$\frac{\mu_F(n_x)}{n_x^{k-1}} \leq \exp\left(-c_{15}\frac{\sqrt{x}}{\log x}\right) \leq \exp\left(-c_4\frac{\sqrt{\log n_x}}{\log\log n_x}\right),$$

where $c_{15} > 0$ is an absolute constant. Thus for given any natural number M, there exists a natural number n with $n > M$ such that

$$\frac{\mu_F(n)}{n^{k-1}} \leq \exp\left(-c_4\frac{\sqrt{\log n}}{\log\log n}\right).$$

Hence we have the result.

5.4.6 Proof of Theorem 5.2.7

Recall that for any $m \in \mathbb{N}$ and any prime p, by (5.3.1) we have

$$\frac{\mu_F(p^m)}{p^{m(k-1)}} = 1 + \frac{1}{p} + \left(1 + \frac{1}{p}\right)\sum_{\ell=1}^{m-1}\frac{a(p^\ell)}{p^{\ell(k-1)}} + \frac{a(p^m)}{p^{m(k-1)}}.$$

Note that the series
$$\sum_{\ell=1}^{\infty}\frac{a(p^\ell)}{p^{\ell(k-1)}}$$
is absolutely convergent (see section 5.4.1 for details). This implies that the sequence

$$\left\{\frac{\mu_F(p^m)}{p^{m(k-1)}}\right\}_{m\in\mathbb{N}}$$

is convergent. Further, there exist absolute constants $e_1, e_2 > 0$ such that

$$1 + \frac{e_1}{p^{1/2}} \leq \frac{\mu_F(p^m)}{p^{m(k-1)}} \leq 1 + \frac{e_2}{p^{1/2}} \tag{5.4.4}$$

holds for all but finitely many primes $p \in A$. Indeed, the upper bound is a consequence of subsection 5.4.1 whereas the lower bound follows from the fact that $\alpha_p \leq 6/p$ for $p \geq 11$ and primes $p \in A$ has the property that $a(p) > \beta \cdot p^{k-3/2}$ with absolute constant β (see subsection 5.4.1, subsection 5.4.2 and subsection 5.4.3).

Let us choose a prime $p_1 \in A$ such that (5.4.4) holds. Since (5.4.4) is true for all but finitely many $p \in A$, we can choose $p_2 \in A$ such that $p_2 > p_1$ and

$$1 + \frac{e_1}{p_2^{1/2}} \;\leq\; \frac{\mu_F(p_2^m)}{p_2^{m(k-1)}} \;\leq\; 1 + \frac{e_2}{p_2^{1/2}} \;<\; 1 + \frac{e_1}{2p_1^{1/2}}.$$

Proceeding in this way, we get a sequence $\{p_n\}_{n\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \frac{\mu_F(p_n^m)}{p_n^{m(k-1)}} \;>\; 1 \quad \text{and} \quad \lim_{m\to\infty} \frac{\mu_F(p_i^m)}{p_i^{m(k-1)}} \;\neq\; \lim_{m\to\infty} \frac{\mu_F(p_j^m)}{p_j^{m(k-1)}}$$

for any $i \neq j$. This shows that there are infinitely many limit points of the sequence $\{\mu_F(n)/n^{k-1}\}_{n\in\mathbb{N}}$ which are in the interval $(1, \infty)$.

Considering the set B (see subsection 5.4.5) and arguing as above, one can show that there is a sequence $\{p_n\}_{n\in\mathbb{N}} \subset B$ for which

$$\lim_{m\to\infty} \frac{\mu_F(p_n^m)}{p_n^{m(k-1)}} \;<\; 1 \quad \text{and} \quad \lim_{m\to\infty} \frac{\mu_F(p_i^m)}{p_i^{m(k-1)}} \;\neq\; \lim_{m\to\infty} \frac{\mu_F(p_j^m)}{p_j^{m(k-1)}}$$

for any $i \neq j$. This completes the proof.

Chapter 6

Hecke eigenvalues of Siegel modular forms of degree two: II

6.1 Introduction

Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms with Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. In this chapter, we investigate arithmetic properties of the sequence $\{\mu_F(n)\mu_G(n)\}_{n\in\mathbb{N}}$ when both F and G lie in the orthogonal complement of Maass subspace. Unlike elliptic modular forms, it is not known whether Hecke eigenforms F and G with $F \neq cG$ for some constant $c \in \mathbb{C}^\times$ implies that $\mu_F(n) \neq \mu_G(n)$ for some $n \in \mathbb{N}$ (see [11, 86], see also [7] for recent progress). This phenomenon is known as *multiplicity one theorem* and it is known to be true under *generalized Böcherer's conjecture* (see [86]). On the other hand, it is known (see [103]) that the *Maass subspace* S_k^* of $S_k(\Gamma_2)$ is generated by the *Saito-Kurokawa lifts* of Hecke eigenforms which are cuspidal elliptic modular forms of weight $2k-2$. Since *multiplicity one theorem* is known to be true for elliptic modular forms and the *Maass subspace* S_k^* is isomorphic to the space $S_{2k-2}(\Gamma_1)$, hence *multiplicity one theorem* holds good for the forms lying in the *Maass subspace* S_k^*.

Henceforth, we shall assume that F and G are Siegel Hecke eigenforms of degree two which do not lie in the *Maass subspace* S_k^*. We also assume that they lie in different eigenspaces. Under this hypothesis, we first investigate the number of positive integers n such that $\mu_F(n) \neq \mu_G(n)$. We then investigate the questions of simultaneous non-vanishing and sign changes of the sequence $\{\mu_F(n)\mu_G(n)\}_{n\in\mathbb{N}}$. The content of this chapter is taken from a joint work with Gun and Kohnen [26].

6.2 Statement of the theorems

We first show the following theorem.

Theorem 6.2.1. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace and having eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. Also let F and G lie in different eigenspaces. Then for any $\epsilon > 0$, one has*

$$\#\{n \le x \mid \mu_F(n) \neq \mu_G(n)\} \;\gg\; x^{1-\epsilon},$$

where the constant $\gg$ depends on F, G and ϵ.

In particular, this theorem shows that at least one of F or G has infinitely many non-zero Hecke eigenvalues. This motivates us to investigate the non-vanishing of Hecke eigenvalues at prime powers. Here we have the following theorem.

Theorem 6.2.2. *Let $F \in S_{k_1}(\Gamma_2)$, $G \in S_{k_2}(\Gamma_2)$, $\mu_F(n)$ and $\mu_G(n)$ be as in Theorem 6.2.1. Then for any prime p, there exists an integer n with $1 \le n \le 14$ such that*

$$\mu_F(p^n)\mu_G(p^n) \;\neq\; 0.$$

Next we investigate the question of Hecke eigenvalues which are of different sign.

More precisely, we have the following theorem.

Theorem 6.2.3. *Let $F \in S_{k_1}(\Gamma_2)$ be a Hecke eigenform lying in the orthogonal complement of the Maass subspace and having Hecke eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$. Also assume that there exist $0 < c < 4$ and a Hecke eigenform $G \in S_{k_2}(\Gamma_2)$ lying in the orthogonal complement of the Maass subspace with Hecke eigenvalues $\{\mu_G(n)\}_{n\in\mathbb{N}}$ such that*

$$\#\left\{p \leq x \mid |\mu_G(p)| > cp^{k_2-\frac{3}{2}}\right\} \geq \frac{16}{17} \cdot \frac{x}{\log x} \tag{6.2.1}$$

for sufficiently large x. If F and G lie in different eigenspaces, then half of the non-zero coefficients of the sequence $\{\mu_F(n)\mu_G(n)\}_{n\in\mathbb{N}}$ are positive and half of them are negative.

First note that the subset $\{p \mid \mu_G(p) = 0\}$ of primes has natural density zero (see appendix of [85]). Further, the *generalized Ramanujan-Petersson* bound implies that for any prime p, we have $|\mu_G(p)| \leq 4p^{k_2-\frac{3}{2}}$. Thus the hypothesis in (6.2.1) is not an unreasonable one (especially if one believes an analogous *Sato-Tate conjecture* in this setup). If we restrict to eigenvalues at primes, we have the following theorem.

Theorem 6.2.4. *Let $F \in S_{k_1}(\Gamma_2)$, $G \in S_{k_2}(\Gamma_2)$, $\mu_F(n)$ and $\mu_G(n)$ be as in Theorem 6.2.3. Then there exists a set of primes p of positive lower density such that $\mu_F(p)\mu_G(p) \gtrless 0$.*

6.3 Some requisites

In order to study analytic properties of $L(F, G; s)$, we shall make use of the following result on the formal power series by Gun and R. Murty [29, Theorem 2].

Theorem 6.3.1. *Let $P_i(T)$ and $Q_i(T)$ be non-zero polynomials over $\mathbb{C}$ such that* *degree of P_i is strictly less than the degree of Q_i for $i = 1, 2$. Also let*

$$Q_1(T) \;:=\; \prod_{i=1}^{r}(1-\alpha_i T)^{\ell_i} \qquad \text{and} \qquad Q_2(T) \;:=\; \prod_{j=1}^{t}(1-\beta_j T)^{m_j},$$

where α_i's are distinct for $1 \le i \le r$ and β_j's are distinct for $1 \le j \le t$ and $\ell_i, m_j \in \mathbb{N}$. Let us also assume that

$$\sum_{n\ge 0} a_n T^n \;=\; \frac{P_1(T)}{Q_1(T)} \qquad \text{and} \qquad \sum_{n\ge 0} b_n T^n \;=\; \frac{P_2(T)}{Q_2(T)}$$

where $a_n, b_n \in \mathbb{C}$ for all $n \ge 0$. Then we have

$$\sum_{n\ge 0} a_n b_n T^n \;=\; \frac{R(T)}{\prod_{i,j}(1-\alpha_i\beta_j T)^{\ell_i m_j}},$$

where $R(T) \in \mathbb{C}[T]$. Now if $a_0 = 1 = b_0$, then $R(0) = 1$. Further if we have $P_1'(0) = 0 = P_2'(0)$, then $R'(0) = 0$. Here P' denotes the derivative of $P(T)$ with respect to T.

To prove Theorem 6.2.3, we will need the following result on the sign changes of multiplicative functions by Matomäki and Radziwiłł [59, Lemma 2.4].

Lemma 6.3.2. *Let $K, L : \mathbb{R}_+ \to \mathbb{R}_+$ be functions such that $K(x) \to 0$ and $L(x) \to \infty$ as $x \to \infty$. Let $g : \mathbb{N} \to \mathbb{R}$ be a multiplicative function such that for every $x \ge 2$, we have*

$$\sum_{\substack{p\ge x,\\ g(p)=0}} \frac{1}{p} \;\le\; K(x) \quad \text{and} \quad \sum_{\substack{p\le x,\\ g(p)<0}} \frac{1}{p} \;\ge\; L(x).$$

Then we have

$$\begin{aligned}\#\big\{n \le x \mid g(n) > 0\big\} \;&=\; (1+o(1)) \cdot \#\big\{n \le x \mid g(n) < 0\big\} \\ &=\; \left(\frac{1}{2} + o(1)\right) x \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p}\right)\left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots\right),\end{aligned}$$

where h is the characteristic function of the set $\{n \in \mathbb{N} \mid g(n) \ne 0\}$.

Let us recall that for any Hecke eigenform $F \in S_k(\Gamma_2)$ lying in the orthogonal complement of the Maass subspace and having Hecke eigenvalues $\mu_F(n)$, we normalized the Hecke eigenvalues as follows:

$$\lambda_F(n) \ := \ \frac{\mu_F(n)}{n^{k-3/2}} \quad \text{for any} \quad n \in \mathbb{N}.$$

We shall call $\lambda_F(n)$ as normalized Hecke eigenvalue.

6.4 Proof of the Theorems

6.4.1 Proof of Theorem 6.2.1

In this subsection, we shall complete the proof of Theorem 6.2.1. Let us start with the following proposition.

Proposition 6.4.1. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms lying in the orthogonal complement of the Maass subspace and having normalized Hecke eigenvalues $\{\lambda_F(n)\}_{n\in\mathbb{N}}$ and $\{\lambda_G(n)\}_{n\in\mathbb{N}}$ respectively. Also let F and G lie in different eigenspaces. Then for sufficiently large x and any $\epsilon > 0$, one has*

$$\sum_{m\le x} \lambda_F(m)\lambda_G(m) \ \ll_\epsilon \quad max\{k_1, k_2\}^{3/8} x^{31/32+\epsilon},$$

where the constant in $\ll_\epsilon$ depends only on ϵ.

To prove Proposition 6.4.1, we first establish a relation between the functions $L(F, G; s)$ and $L(F \times G, s)$. More precisely, we show the following.

Lemma 6.4.2. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Proposition 6.4.1. Then for $\Re(s) > 1$, one has*

$$L(F, G; s) \ = \ g(s)L(F \times G; s), \tag{6.4.1}$$

where

$$g(s) \;:=\; \prod_{p\in\mathcal{P}} g_p(p^{-s}). \tag{6.4.2}$$

Here $g_p(X)$'s are polynomials of degree ≤ 15 and the Euler product on the right hand side of (6.4.2) *is absolutely convergent for $\Re(s) > 1/2$. Further, there exists an absolute constant $A > 0$ such that*

$$g(s) \;\ll\; \sigma^A \left(\sigma - \frac{1}{2}\right)^{-A}$$

holds uniformly for any $\sigma := \Re(s) > 1/2$.

Proof of Lemma 6.4.2. Consider the L-functions

$$L(F,s) \;:=\; \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} \qquad \text{and} \qquad L(G,s) \;:=\; \sum_{n=1}^{\infty} \frac{\lambda_G(n)}{n^s}.$$

These L-functions are absolutely convergent for $\Re(s) > 1$ and by (2.2.2), we have

$$L(F,s) \;=\; \frac{Z_F(s)}{\zeta(2s+1)} \qquad \text{and} \qquad L(G,s) \;=\; \frac{Z_G(s)}{\zeta(2s+1)}.$$

Here $Z_F(s), Z_G(s)$ are the spinor zeta functions associated to F and G respectively. Since $\lambda_F(n)$ and $\lambda_G(n)$ are multiplicative, for any prime p, by (2.2.3), we can write

$$\sum_{n=0}^{\infty} \lambda_F(p^n)T^n \;=\; \frac{1-\frac{1}{p}T^2}{\prod_{1\le i\le 4}(1-\alpha_{p,i}T)} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \lambda_G(p^n)T^n \;=\; \frac{1-\frac{1}{p}T^2}{\prod_{1\le i\le 4}(1-\beta_{p,i}T)}.$$

Now by Theorem 6.3.1, one has

$$\sum_{n=0}^{\infty} \lambda_F(p^n)\lambda_G(p^n)T^n \;=\; \frac{g_p(T)}{\prod_{1\le i,j\le 4}(1-\alpha_{p,i}\beta_{p,j}T)},$$

where $g_p(T) \in \mathbb{C}[T]$ is a polynomial of degree at most 15. Also $g_p(0) = 1$ and

$g_p'(0) = 0$, where g_p' is the derivative of g_p. The fact $|\alpha_{p,i}| = |\beta_{p,j}| = 1$ for $1 \leq i, j \leq 4$ implies that the coefficients of $g_p(T)$ are bounded by an absolute constant. Since $g_p(0) = 1$, the coefficients of T in the polynomial $g_p(T)$ is zero and other coefficients are bounded by an absolute constant, it is easy to conclude that

$$\prod_{p \in \mathcal{P}} g_p(p^{-s})$$

is absolutely convergent for $\Re(s) > 1/2$. This shows that for $\sigma > 1$, we have

$$L(F, G; s) \;=\; L(F \times G; s) g(s).$$

It remains to show that $g(s)$ has the required bound. Let

$$g_p(T) \;:=\; 1 + a(p^2)T^2 + \cdots + a(p^{15})T^{15},$$

where $a(p^i) \in \mathbb{C}$ and $a(p^i)$ are bounded by an absolute constant for all $2 \leq i \leq 15$ and for all primes p. Let $A > 0$ be an integer such that $|a(p^2)| \leq A$ for all $p \in \mathcal{P}$. Thus

$$|g_p(p^{-s})| \;=\; \left| 1 + \sum_{2 \leq n \leq 15} a(p^n) p^{-ns} \right| \;\leq\; h_p(\sigma),$$

where

$$h_p(s) \;:=\; 1 + Ap^{-2s} + |a(p^3)| p^{-3s} + \cdots + |a(p^{15})| p^{-15s}.$$

Now note that

$$\left(1 - p^{-2s}\right)^A h_p(s) \;=\; 1 + O\left(p^{-3\sigma}\right). \tag{6.4.3}$$

The left hand side of (6.4.3) is nothing but the p-th Euler factor of the Dirichlet series

$$\zeta(2s)^{-A} h(s), \qquad \text{where} \qquad h(s) \;:=\; \prod_{p \in \mathcal{P}} h_p(s).$$

Hence for all $\sigma > 1/2$, we have

$$g(s) \ll \left(\frac{\sigma}{\sigma - 1/2}\right)^A.$$

This completes the proof of Lemma 6.4.2. $\square$

As an application of the above lemma, one can derive the following analytic properties of the L-function $L(F, G; s)$.

Lemma 6.4.3. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Proposition 6.4.1. Then the function $L(F, G; s)$ admits an analytic continuation to $\Re(s) > 1/2$.*

Proof of Lemma 6.4.3. We know from Lemma 6.4.2 that for any $\sigma > 1$, we have

$$L(F, G; s) = g(s) L(F \times G, s).$$

Now holomorphicity of $g(s)$ to $\Re(s) > 1/2$ along with the fact that $L(F \times G, s)$ has analytic continuation to $\mathbb{C}$ (see Theorem 2.2.10) imply that $L(F, G; s)$ can be continued analytically upto $\sigma > 1/2$. $\square$

To prove Proposition 6.4.1, we also need the following convexity bound.

Lemma 6.4.4. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be as in Proposition 6.4.1. Then for any $\epsilon > 0$ and $0 < \delta < 1$, one has*

$$L(F \times G, \delta + it) \ll_\epsilon max\{k_1, k_2\}^{6(1-\delta+\epsilon)} |3 + it|^{8(1-\delta+\epsilon)}. \tag{6.4.4}$$

To prove Lemma 6.4.4 we shall use the following strong convexity principle due to Rademacher [76] which we recall here.

Proposition 6.4.5. *Let $g(s)$ be holomorphic and of finite order in $a < \Re(s) < b$,* *and continuous on the closed strip $a \leq \Re(s) \leq b$. Also let*

$$|g(a+it)| \ \leq \ E|P+a+it|^{\alpha} \quad \text{and} \quad |g(b+it)| \ \leq \ F|P+b+it|^{\beta},$$

where E, F are positive constants and P, α, β are real constants satisfying

$$P+a \ > \ 0, \quad \alpha \ \geq \ \beta.$$

Then for $a < \sigma < b$, we have

$$|g(s)| \ \leq \ \left(E|P+s|^{\alpha}\right)^{\frac{b-\sigma}{b-a}} \left(F|P+s|^{\beta}\right)^{\frac{\sigma-a}{b-a}}.$$

We now complete the proof of Lemma 6.4.4.

Proof of Lemma 6.4.4. Without loss of generality, let us assume that $k_1 \geq k_2 > 2$. It is known by [74, sec. 5.1] that F (also G) can be associated to a cuspidal, automorphic representation π (resp. π') of $\mathrm{GSp}_4(\mathbb{A})$ such that π (resp. π') has trivial central character, the archimedean component π_∞ (resp. π'_∞) is a holomorphic discrete series representation with scalar minimal K-type (k_1, k_1) [resp. (k_2, k_2)] and for each finite place p, the local representation π_p [resp. π'_p] is unramified. Here $\mathbb{A}$ is the ring of adeles of $\mathbb{Q}$. The real Weil group $W_{\mathbb{R}}$ is given by $\mathbb{C}^\times \sqcup j\mathbb{C}^\times$ such that $j^2 = -1$ and $jzj^{-1} = \overline{z}$ for $z \in \mathbb{C}^\times$. Then the real Weil group representations underlying Siegel modular forms F and G of weights k_1 and k_2 respectively are given by (see page 90 of [74] and page 2397 of [88]) $\varphi_{2k_1-3} \oplus \varphi_1$ and $\varphi_{2k_2-3} \oplus \varphi_1$, where for $k \in \mathbb{N}$, φ_k is defined by

$$\varphi_k : \mathbb{C}^\times \ni re^{i\theta} \mapsto \begin{bmatrix} e^{ik\theta} & \\ & e^{-ik\theta} \end{bmatrix}, \quad j \mapsto \begin{bmatrix} & (-1)^k \\ 1 & \end{bmatrix}.$$

Then the parameter of $\pi_\infty \times \pi'_\infty$ is

$$(\varphi_{2k_1-3} \oplus \varphi_1) \otimes (\varphi_{2k_2-3} \oplus \varphi_1) \;=\; \begin{cases} \varphi_{2k_1+2k_2-6} \oplus \varphi_{2(k_1-k_2)} \oplus \varphi_{2k_1-2} & \\ \qquad \oplus\, \varphi_{2k_1-4} \oplus \varphi_{2k_2-2} \oplus \varphi_{2k_2-4} & \\ \qquad \oplus\, \varphi_2 \oplus \varphi_+ \oplus \varphi_- & \text{if } k_1 > k_2 \\ \varphi_{4k_1-6} \oplus \varphi_+ \oplus \varphi_- \oplus \varphi_{2k_1-2} & \\ \qquad \oplus\, \varphi_{2k_1-4} \oplus \varphi_{2k_1-2} \oplus \varphi_{2k_1-4} & \\ \qquad \oplus\, \varphi_2 \oplus \varphi_+ \oplus \varphi_- & \text{if } k_1 = k_2. \end{cases}$$

Here φ_+ and φ_- are given by

$$\varphi_+ : re^{i\theta} \mapsto 1, \quad j \mapsto 1;$$
$$\varphi_- : re^{i\theta} \mapsto 1, \quad j \mapsto -1.$$

Now from [88, Table 2], one can easily see that the gamma factors of $L(F \times G, s)$ are as follows:

$$L_\infty(F \times G, s) \;:=\; \begin{cases} \Gamma_{\mathbb{C}}(s + k_1 + k_2 - 3)\Gamma_{\mathbb{C}}(s + k_1 - k_2)\Gamma_{\mathbb{C}}(s + k_1 - 1) & \\ \qquad \Gamma_{\mathbb{C}}(s + k_1 - 2)\Gamma_{\mathbb{C}}(s + k_2 - 1)\Gamma_{\mathbb{C}}(s + k_2 - 2) & \\ \qquad \Gamma_{\mathbb{C}}(s + 1)\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 1) & \text{if } k_1 > k_2, \\ \Gamma_{\mathbb{C}}(s + 2k_1 - 3)\Gamma_{\mathbb{C}}^2(s + k_1 - 1)\Gamma_{\mathbb{C}}^2(s + k_1 - 2) & \\ \qquad \Gamma_{\mathbb{C}}(s + 1)\Gamma_{\mathbb{R}}^2(s)\Gamma_{\mathbb{R}}^2(s + 1) & \text{if } k_1 = k_2, \end{cases}$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. Again by [74, Theorem 5.2.3], we know that the completed L-function

$$L^*(F \times G, s) \;:=\; L_\infty(F \times G, s)L(F \times G, s)$$

satisfies the functional equation

$$L^*(F \times G, 1-s) \;=\; \epsilon(F \times G, s) L^*(F \times G, s),$$

where $\epsilon(F \times G, s) \in \mathbb{C}$ and has absolute value 1. Thus for any $s \in \mathbb{C}$ with $\sigma > 1$, we have

$$|L(F \times G, 1-s)| \;=\; \left| \frac{L_\infty(F \times G, s)}{L_\infty(F \times G, 1-s)} \right| \cdot |L(F \times G, s)| \,.$$

Note that for $s = c + it$ with $1 < c < 3/2$, we have

$$\left| \frac{L_\infty(F \times G, c+it)}{L_\infty(F \times G, 1-c-it)} \right| \;\ll\; k_1^{6(2c-1)} |1+it|^{8(2c-1)}.$$

Let $c = 1 + \epsilon$ with $0 < \epsilon < 1/2$. Since $|L(F \times G, 1+\epsilon+it)| \ll_\epsilon 1$, for any $0 < \delta < 1$, using Proposition 6.4.5, we have

$$|L(F \times G, \delta + it)| \;\ll\; k_1^{6(1-\delta+\epsilon)} |3+it|^{8(1-\delta+\epsilon)}.$$

This completes the proof of the lemma. $\square$

Now we are ready to prove Proposition 6.4.1.

Proof of Proposition 6.4.1. From the work of Weissauer [99] one knows that the generalized Ramanujan-Petersson conjecture is true for F and G, that is, for any $\epsilon > 0$, one has

$$\lambda_F(n)\lambda_G(n) \;\ll\; n^\epsilon.$$

Hence by the Perron's summation formula, we have

$$\sum_{n \le x} \lambda_F(n)\lambda_G(n) \;=\; \frac{1}{2\pi i} \int_{1+\epsilon-iT}^{1+\epsilon+iT} L(F, G; s) \frac{x^s}{s} ds \;+\; O\left(\frac{x^{1+2\epsilon}}{T} \right).$$

Now we shift the line of integration to $1/2 < \Re(s) := \delta < 1$ (to be chosen later).

Since there are no singularities of the function $L(F,G;s)x^s/s$ in the region bounded by the lines joining the points $1+\epsilon-iT, 1+\epsilon+iT, \delta+iT$ and $\delta-iT$, we have

$$\sum_{n\le x}\lambda_F(n)\lambda_G(n) \;=\; I_1+I_2+I_3 \;+\; O\left(\frac{x^{1+2\epsilon}}{T}\right),$$

where

$$I_1 \;:=\; \frac{1}{2\pi i}\int_{\delta-iT}^{\delta+iT} L(F,G;s)\frac{x^s}{s}ds, \qquad I_2 := \frac{1}{2\pi i}\int_{\delta+iT}^{1+\epsilon+iT} L(F,G;s)\frac{x^s}{s}ds$$
$$\text{and} \qquad I_3 \;:=\; \frac{1}{2\pi i}\int_{1+\epsilon-iT}^{\delta-iT} L(F,G;s)\frac{x^s}{s}ds.$$

Using Lemma 6.4.2 and Lemma 6.4.4, one can easily get

$$I_1 \;\ll_\epsilon\; (\delta-1/2)^{-A}k^{6(1-\delta+\epsilon)}x^{\delta}T^{8(1-\delta+\epsilon)},$$

where $k = \max(k_1,k_2)$. Similarly, one can get

$$I_2, I_3 \;\ll_\epsilon\; (\delta-1/2)^{-A}k^{6(1-\delta+\epsilon)}x^{1+\epsilon}T^{8(1-\delta+\epsilon)-1}.$$

We shall put $T = x^{\alpha}$, where $\alpha > 0$ is a real number (to be chosen later). Thus we have

$$\sum_{n\le x}\lambda_F(n)\lambda_G(n) \;\ll_\epsilon\; (\delta-1/2)^{-A}k^{6(1-\delta+\epsilon)}\left(x^{8\alpha(1-\delta+\epsilon)+\delta}+x^{1+8\alpha(1-\delta+\epsilon)-\alpha+\epsilon}+x^{1-\alpha+\epsilon}\right).$$

Choosing $\alpha = 1/16$ and $\delta = 15/16$, one has

$$\sum_{n\le x}\lambda_F(n)\lambda_G(n) \;\ll_\epsilon\; k^{3/8+\epsilon}x^{31/32+\epsilon}.$$

This completes the proof of Proposition 6.4.1. □

Proof of Theorem 6.2.1. We know from Theorem 2.2.11 and Proposition 6.4.1

that

$$\sum_{n\leq x}\lambda_F^2(n) \;=\; c_F x + O(x^{\frac{31}{32}}) \quad\text{and}\quad \sum_{n\leq x}\lambda_F(n)\lambda_G(n) \;=\; O(x^{\frac{31}{32}}),$$

where $c_F > 0$. Suppose that $k_2 \leq k_1$. Using partial summation, we get

$$\text{(6.4.5)} \qquad \begin{aligned} \sum_{n\leq x}\mu_F^2(n) \;&=\; c x^{2k_1-2} + O\left(x^{2k_1-2-\frac{1}{32}}\right) \\ \text{and} \qquad \sum_{n\leq x}\mu_F(n)\mu_G(n) \;&=\; O\left(x^{k_1+k_2-2-\frac{1}{32}}\right), \end{aligned}$$

where $c = c_F/2k_1 - 2$. Now let

$$S(x) \;:=\; \sum_{n\leq x}\big[\mu_F(n) - \mu_G(n)\big]\mu_F(n).$$

Note that for any $\epsilon > 0$, we have

$$S(x) \;\leq\; c(\epsilon)\cdot \#\left\{n \in \mathbb{N} \mid n \leq x, \mu_F(n) \neq \mu_G(n)\right\} x^{2k_1-3+\epsilon},$$

where $c(\epsilon) > 0$ is a constant depending only on $\epsilon > 0$. Now by applying (6.4.5), we conclude that

$$\#\left\{n \in \mathbb{N} \mid n \leq x, \mu_F(n) \neq \mu_G(n)\right\} \;\;\gg_{F,G,\epsilon}\;\; x^{1-\epsilon}.$$

When $k_1 \leq k_2$, we consider the sum $\sum_{n\leq x}[\mu_G(n) - \mu_F(n)]\mu_G(n)$ and proceed as above to get the result. This completes the proof of Theorem 6.2.1.

Remark 6.4.6. *To prove Theorem 6.2.1, we have only used the property*

$$\sum_{n\leq x}\lambda_F(n)\lambda_G(n) \;=\; o(x),$$

as $x \to \infty$ but Proposition 6.4.1 gives an explicit upper bound and hence it is of independent interest.

6.4.2 Proof of Theorem 6.2.2

In this subsection, we shall give a proof of Theorem 6.2.2. We start by recalling that for any prime p and any natural number $n \geq 3$, one has

$$(6.4.6) \qquad \begin{aligned} \lambda_F(p^n) &= \lambda_F(p)\lambda_F(p^{n-1}) - \left[\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p}\right]\lambda_F(p^{n-2}) \\ &\quad + \lambda_F(p)\lambda_F(p^{n-3}) - \lambda_F(p^{n-4}), \end{aligned}$$

with the assumption that $\lambda_F(p^{n-m}) = 0$ for $n < m$ are natural numbers. Similar relations hold among the Hecke eigenvalues $\lambda_G(p^n)$ for $n \geq 3$. We use these relations to derive some important consequences which will help us to prove our result. We start with a general result which might be of independent interest.

Lemma 6.4.7. *Let $f_0(x) = -1$ and $f_1(x) = -x$ be polynomials over $\mathbb{Z}$. Define a family of polynomials $\{f_n\}_{n\in\mathbb{N}}$ by*

$$(6.4.7) \qquad f_{n+1}(x) = xf_n(x) - f_{n-1}(x).$$

Then for any $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$, we have $f_n(\alpha) \neq 0$ for all $n \in \mathbb{N}$.

Proof of Lemma 6.4.7. We first show by induction on $n \in \mathbb{N}$ that

$$(6.4.8) \qquad f_n(x) = -x^n + a_{n,n-1}x^{n-1} + a_{n,n-2}x^{n-2} + \cdots + a_{n,1}x + a_{n,0},$$

where $a_{n,i} \in \mathbb{Z}$ for $0 \leq i \leq n-2$. Note that this is true for $n = 0, 1$. Using (6.4.7), we get

$$f_{n+1}(x) = -x^{n+1} + a_{n,n-1}x^n + (a_{n,n-2}+1)x^{n-1} + \cdots + (a_{n,0} - a_{n-1,1})x - a_{n-1,0}.$$

Hence by induction we have (6.4.8). Since $\mathbb{Z}$ is integrally closed, any solution in $\mathbb{Q}$ of $f_n(x)$ for any n will be an integer. This completes the proof of the lemma. $\square$

Lemma 6.4.8. *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform which lies in the orthogonal complement of the Maass subspace with normalized Hecke eigenvalues $\lambda_F(n)$ for $n \in \mathbb{N}$. Then*

1. *If $\lambda_F(p^{2m}) = 0$ for some $m \geq 2$, then at least one of $\lambda_F(p), \lambda_F(p^2)$ is non-zero.*
2. *There does not exist $t \in \mathbb{N}$ such that*

$$\lambda_F(p^m) \;=\; 0 \quad \textit{for} \quad t+1 \leq m \leq t+4.$$

Proof of Lemma 6.4.8. Suppose that $\lambda_F(p) = 0 = \lambda_F(p^2)$. Then for any $n \geq 0$,

$$\lambda_F(p^{2n+4}) \;=\; f_n\left(\frac{1}{p}\right),$$

where f_n's are polynomials in $\mathbb{Z}[x]$ satisfying the hypothesis of Lemma 6.4.7. Hence by Lemma 6.4.7, we have $\lambda_F(p^{2m}) \neq 0$ for all $m \geq 2$, a contradiction to our hypothesis. This completes the proof of the first part of the lemma.

To prove the second part of the lemma, let us assume that $\lambda_F(p^m) = 0$ for $t+1 \leq m \leq t+4$. Using (6.4.6), we have

$$\lambda_F(p^t) \;=\; -\lambda_F(p^{t+4}) \;=\; 0.$$

Using induction and the identity (6.4.6), we get that $\lambda_F(p^m) = 0$ for $1 \leq m \leq t+4$. This implies that $\lambda_F(p) = 0 = \lambda_F(p^2)$, a contradiction to the first part of the lemma. $\square$

Lemma 6.4.9. *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform which lies in the orthogonal complement of the Maass subspace with normalized Hecke eigenvalues $\lambda_F(n)$ for $n \in \mathbb{N}$. Then*

1. *For some $m \geq 0$, $\lambda_F(p^{2m+1}) \neq 0$ implies that $\lambda_F(p) \neq 0$.*

2. If $\lambda_F(p) \neq 0$, then for any $m \in \mathbb{N}$, there exists $0 \leq i \leq 3$ such that $\lambda_F(p^{2(m+i)+1}) \neq 0$.

Proof of Lemma 6.4.9. We shall show by induction on m that $\lambda_F(p) = 0$ implies that $\lambda_F(p^{2m+1}) = 0$ for all $m \geq 0$. It is clearly true for $m = 0, 1$. Using (6.4.6), we get

$$\lambda_F(p^{2m+1}) = \left[\lambda_F(p^2) + \frac{1}{p}\right] \lambda_F(p^{2m-1}) - \lambda_F(p^{2m-3}).$$

By induction hypothesis, one knows that

$$\lambda_F(p^{2m-1}) = 0 = \lambda_F(p^{2m-3})$$

and hence $\lambda_F(p^{2m+1}) = 0$. This completes the proof of the first part.

To prove the second part, assume that there exist $m_0 \in \mathbb{N}$ such that

$$\lambda_F(p^{2(m_0+i)+1}) = 0 \tag{6.4.9}$$

for all $0 \leq i \leq 3$. Using (6.4.6) and (6.4.9) for $i = 2, 3$, we have

$$\lambda_F(p^{2m_0+6}) = -\lambda_F(p^{2m_0+4}) = \lambda_F(p^{2m_0+2})$$

as $\lambda_F(p) \neq 0$. Again using (6.4.6) and (6.4.9), we get

$$\lambda_F(p^{2m_0+6}) = -\left[\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p}\right] \lambda_F(p^{2m_0+4}) - \lambda_F(p^{2m_0+2}).$$

Hence

$$\begin{aligned} 0 = \lambda_F(p^{2m_0+6}) + \lambda_F(p^{2m_0+4}) &= -\left[\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} - 1\right] \lambda_F(p^{2m_0+4}) \\ &\qquad - \lambda_F(p^{2m_0+2}) \\ &= -\left[\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} - 2\right] \lambda_F(p^{2m_0+2}). \end{aligned}$$

This implies that

$$\lambda_F^2(p) - \lambda_F(p^2) - \frac{1}{p} \;=\; 2$$

as $\lambda_F(p^{2m_0+2}) \neq 0$ by second part of Lemma 6.4.8. Replacing

$$\lambda_F(p^{2m_0+4}) \;=\; -\,2\lambda_F(p^{2m_0+2}) - \lambda_F(p^{2m_0})$$

in the relation

$$0 = \lambda_F(p^{2m_0+5}) \;=\; \lambda_F(p)\left[\lambda_F(p^{2m_0+4}) + \lambda_F(p^{2m_0+2})\right],$$

we get $\lambda_F(p^{2m_0+2}) + \lambda_F(p^{2m_0}) = 0$ as $\lambda_F(p) \neq 0$. Then

$$0 = \lambda_F(p^{2m_0+3}) \;=\; \lambda_F(p)\left[\lambda_F(p^{2m_0+2}) + \lambda_F(p^{2m_0})\right] - \lambda_F(p^{2m_0-1}) \;=\; -\,\lambda_F(p^{2m_0-1}).$$

This shows that if $\lambda_F(p) \neq 0$ and $\lambda_F(p^{2(m_0+i)+1}) = 0$ for some $m_0 \in \mathbb{N}$ and for all $0 \leq\; i \leq\; 3$, then $\lambda_F(p^{2m_0-1}) = 0$. Arguing similarly and using induction, we can now show that $\lambda_F(p^{2m+1}) = 0$ for all $1 \leq m \leq m_0 + 3$. Note that

$$\begin{aligned} 0 \;=\; \lambda_F(p^5) &= \lambda_F(p)\big[\lambda_F(p^4) + \lambda_F(p^2) - 1\big] \\ &= \lambda_F(p)\big[-\lambda_F(p^2) + \lambda_F^2(p) - 2\big] \\ &= \frac{1}{p}\lambda_F(p), \end{aligned}$$

a contradiction to our hypothesis. This completes the proof of Lemma 6.4.9. $\square$

Remark 6.4.10. *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform which lies in the orthogonal complement of the Maass subspace with normalized Hecke eigenvalues $\lambda_F(n)$ for $n \in \mathbb{N}$. If $\lambda_F(p) \neq 0$, then there does not exist $m \in \mathbb{N}$ such that $\lambda_F(p^{2(m+i)}) = 0$ for all $0 \leq i \leq 3$.*

Proof of Remark (6.4.10)**.** Suppose that there exists $m_0 \in \mathbb{N}$ such that

$$\lambda_F(p^{2(m_0+i)}) \;=\; 0, \quad \text{for} \quad 0 \le i \le 3.$$

Arguing as in Lemma 6.4.9, then we have $2 + 1/p + \lambda_F(p^2) - \lambda_F^2(p) = 0$ and $\lambda_F(p^{2m}) = 0$ for $1 \le m \le m_0 + 3$ as $\lambda_F(p) \ne 0$. This implies that $\lambda_F^2(p) = 2 + 1/p$ and hence $\lambda_F(p^4) = -1$, a contradiction. □

Proof of Theorem 6.2.2. Without loss of generality, we can assume that $\lambda_F(p)\lambda_G(p) = 0$ and $\lambda_F(p^2)\lambda_G(p^2) = 0$, otherwise we are done.

First suppose that $\lambda_F(p) = \lambda_G(p) = \lambda_F(p^2) = \lambda_G(p^2) = 0$. Then using the identity (6.4.6), we see that $\lambda_F(p^4)\lambda_G(p^4) = 1$. Hence we are done.

Now we assume that $\lambda_F(p) = \lambda_G(p) = \lambda_F(p^2) = 0$ but $\lambda_G(p^2) \ne 0$. Then

$$\lambda_G(p^6) \;=\; \left[\lambda_G(p^2) + \frac{1}{p}\right] \lambda_G(p^4) \;-\; \lambda_G(p^2)$$

implies that either $\lambda_G(p^4) \ne 0$ or $\lambda_G(p^6) \ne 0$. Now using Lemma 6.4.8, we are done.

Next assume that $\lambda_F(p) = 0 = \lambda_F(p^2)$ and $\lambda_G(p) \ne 0$. Using Lemma 6.4.8, we know that $\lambda_F(p^{2n}) \ne 0$ for all $n \ge 2$. Since $\lambda_G(p) \ne 0$, by Remark (6.4.10), we have at least one of

$$\lambda_G(p^4),\ \lambda_G(p^6),\ \lambda_G(p^8),\ \lambda_G(p^{10})$$

is non-zero. Hence we are done in this case.

Finally, we assume that $\lambda_F(p) = 0, \lambda_F(p^2) \ne 0$ and $\lambda_G(p) \ne 0, \lambda_G(p^2) = 0$. Since $\lambda_F(p) = 0$ we know by Lemma 6.4.9 that $\lambda_F(p^{2n-1}) = 0$ for all $n \in \mathbb{N}$.

We first consider the case when $\lambda_F(p^4) = 0$. Then using (6.4.6), we have $\lambda_F(p^n) \ne 0$ for $n = 6, 8, 10, 12$. Since $\lambda_G(p) \ne 0$, using Remark (6.4.10) we are done.

Now assume that $\lambda_F(p^4) \neq 0$ and $\lambda_G(p^4) = 0$, otherwise we are done. We will show in this case that $\lambda_G(p^6) \neq 0$ except when $p = 2$. Since $\lambda_G(p^4) = 0$, we get

$$\left[2 + 1/p - \lambda_G^2(p)\right] \lambda_G^2(p) = 1. \tag{6.4.10}$$

Using (6.4.10) and (6.4.6), we have

$$\begin{aligned} \lambda_G(p^6) &= -\lambda_G^2(p) + \lambda_G(p)\lambda_G(p^3) \left[1 + \frac{1}{p} - \lambda_G^2(p)\right] \\ &= -\lambda_G^2(p) + \lambda_G^2(p^3) \quad = \quad \frac{1}{p} - \lambda_G^2(p). \end{aligned}$$

Again using (6.4.10), we see that $1/p - \lambda_G^2(p) = 0$ only when $p = 2$. If $\lambda_F(p^6) \neq 0$, we are done except when $p = 2$. So without loss of generality, we can assume that $\lambda_F(p^6) = 0$ when $p \neq 2$. Then

$$1 + \lambda_F(p^4) \;=\; \left[\lambda_F(p^2) + \frac{1}{p}\right] \lambda_F(p^2), \quad \lambda_F(p^2) \;=\; \left[\lambda_F(p^2) + \frac{1}{p}\right] \lambda_F(p^4)$$

and hence

$$\lambda_F(p^8) \;=\; -\lambda_F(p^4), \quad \lambda_F(p^{10}) \;=\; -\lambda_F(p^2), \quad \lambda_F(p^{12}) \;=\; -1, \quad \lambda_F(p^{14}) \;=\; -\frac{1}{p}.$$

We are now done by Remark (6.4.10).

It only remains to prove the case when $p = 2$ and $\lambda_G^2(2) = 1/2$. In this case,

$$\lambda_G(2^8) = -1 \quad \text{and} \quad \lambda_G(2^{10}) = -1/2.$$

Now note that either $\lambda_F(2^8) \neq 0$ or $\lambda_F(2^8) = 0$ and $\lambda_F(2^{10}) = -\lambda_F(2^6) \neq 0$. This completes the proof of Theorem 6.2.2. $\square$

6.4.3 An intermediate lemma

Aim of subsection is to prove the following lemma which is required to complete the proof of Theorem 6.2.3 and Theorem 6.2.4.

Lemma 6.4.11. *Let $F \in S_{k_1}(\Gamma_2)$ and $G \in S_{k_2}(\Gamma_2)$ be Hecke eigenforms which do not lie in the Maass subspace and having normalized eigenvalues $\{\lambda_F(n)\}_{n\in\mathbb{N}}$ and $\{\lambda_G(n)\}_{n\in\mathbb{N}}$ respectively. Also assume that F and G lie in different eigenspaces and there exists $0 < c < 4$ such that*

$$\#\{p \le x \mid |\lambda_G(p)| > c\} \;\ge\; \frac{16}{17}\cdot\frac{x}{\log x}$$

for sufficiently large x. Then we have

$$\sum_{p\le x}\lambda_F^2(p)\lambda_G^2(p) \;\gg\; \frac{x}{\log x}.$$

Proof of Lemma 6.4.11. By [74, Theorem 5.1.2], one knows that the transfers of F and G are irreducible unitary cuspidal and self-contragredient automorphic representations of $\mathrm{GL}_4(\mathbb{A})$. Hence by [101, Theorem 3], we have

$$\sum_{p\le x}\lambda_F^2(p) \;=\; \frac{x}{\log x}+o\left(\frac{x}{\log x}\right) \quad \text{and} \quad \sum_{p\le x}\lambda_G^2(p) \;=\; \frac{x}{\log x}+o\left(\frac{x}{\log x}\right),$$

as $x\to\infty$. Let S be the set of primes p such that $|\lambda_G(p)| > c$. Thus for sufficiently large x, we have

$$\sum_{p\le x}\lambda_F^2(p)\lambda_G^2(p) \;>\; c^2\sum_{\substack{p\le x,\\ p\in S}}\lambda_F^2(p).$$

Now by the given hypothesis, one has

$$\sum_{\substack{p\le x,\\ p\notin S}}\lambda_F^2(p) \;\le\; 16\cdot\#\{p\le x \mid p\notin S\} \;\le\; \frac{16}{17}\cdot\frac{x}{\log x}$$

for sufficiently large x. This implies that

$$\sum_{p \le x} \lambda_F^2(p) \lambda_G^2(p) \;\gg\; \frac{x}{\log x}$$

for sufficiently large x. This completes the proof of Lemma 6.4.11. $\square$

In next subsection, we complete the proof of Theorem 6.2.4 and then use Theorem 6.2.4 to complete the proof of Theorem 6.2.3.

6.4.4 Proof of Theorem 6.2.4

Using [74, Theorem 5.1.2], we know that the transfers of F and G are irreducible unitary cuspidal and self-contragredient automorphic representations of $\mathrm{GL}_4(\mathbb{A})$. Hence by [101, Theorem 3], we have

$$\sum_{p \le x} \lambda_F(p) \lambda_G(p) \;=\; o\left(\frac{x}{\log x}\right), \tag{6.4.11}$$

as $x \to \infty$. Consider the sum

$$S^+(x) \;:=\; \sum_{p \le x} \big[\lambda_F(p)\lambda_G(p) + 16\big] \lambda_F(p)\lambda_G(p).$$

Observe that

$$\begin{aligned} S^+(x) \;&\le\; \sum_{\substack{p \le x, \\ \lambda_F(p)\lambda_G(p) > 0}} \big[\lambda_F(p)\lambda_G(p) + 16\big] \lambda_F(p)\lambda_G(p) \\ &\le\; 512 \cdot \#\big\{p \le x \mid \lambda_F(p)\lambda_G(p) > 0\big\}. \end{aligned} \tag{6.4.12}$$

On the other hand, using Lemma 6.4.11 and (6.4.11) for sufficiently large x, we have

$$S^+(x) \;=\; \sum_{p \le x} \lambda_F^2(p)\lambda_G^2(p) \;+\; 16 \sum_{p \le x} \lambda_F(p)\lambda_G(p) \;\gg\; \frac{x}{\log x}. \tag{6.4.13}$$

Thus by (6.4.12) and (6.4.13), we conclude that there exists a set of primes p having positive density such that $\lambda_F(p)\lambda_G(p) > 0$. Similarly, by considering the sum

$$S^-(x) \ := \ \sum_{p\le x} \big[\lambda_F(p)\lambda_G(p) - 16\big]\lambda_F(p)\lambda_G(p)$$

and arguing as above one can conclude that there exists a set of primes p having positive density such that $\lambda_F(p)\lambda_G(p) < 0$. □

6.4.5 Proof of Theorem 6.2.3

It follows from Theorem 2.2.12 that there exists $\delta > 0$ such that

$$\begin{aligned}\#\big\{p \le x \mid \lambda_F(p)\lambda_G(p) = 0\big\} \ &\le \ \#\big\{p \le x \mid \lambda_F(p) = 0\big\} \ + \ \#\big\{p \le x \mid \lambda_G(p) = 0\big\} \\ &= \ O\left(\frac{x}{(\log x)^{1+\delta}}\right)\end{aligned}$$

for sufficiently large x. Also note that by Theorem 6.2.4, the set

$$\{p \in \mathcal{P} \mid \lambda_F(p)\lambda_G(p) < 0\}$$

has positive lower density. Hence the multiplicative function $\lambda_F(n)\lambda_G(n)$ satisfies the hypothesis of Lemma 6.3.2. We now apply Lemma 6.3.2 to complete the proof of Theorem 6.2.3. □

Remark 6.4.12. *Let F, G be non-CM Siegel cusp forms of degree one, of weights k_1, k_2 and levels N_1, N_2 respectively. Also let F and G be distinct Hecke eigenforms with eigenvalues $\{\mu_F(n)\}_{n\in\mathbb{N}}$ and $\{\mu_G(n)\}_{n\in\mathbb{N}}$ respectively. Then the method adopted here for Theorem 6.2.3 can be applied to prove unconditionally that half of the non-zero coefficients of the sequence $\{\mu_F(n)\mu_G(n)\}_{n\in\mathbb{N}}$ are positive and half of them are negative. One can also show unconditionally that there exists a set of primes p of positive lower density such that $\mu_F(p)\mu_G(p) \gtrless 0$.*

Bibliography

[1] E. Alkan and A. Zaharescu, *$\mathfrak{B}$-free numbers in short arithmetic progressions*, J. Number Theory **113** (2005), 226–243.

[2] E. Alkan and A. Zaharescu, *Nonvanishing of the Ramanujan tau function in short intervals*, Int. J. Number Theory **1** (2005), no. 1, 45–51.

[3] A. N. Andrianov, *Euler products that correspond to Siegel modular forms of genus* 2, Russian Math. Surveys **29:3** (1974), 45–116.

[4] A. N. Andrianov, *Modular descent and the Saito-Kurokawa conjecture*, Inv. Math. **53** (1979) 267 – 280.

[5] S. Arias-de-Reyna, I. Inam and G. Wiese, *On conjectures of Sato-Tate and Bruinier-Kohnen*, Ramanujan J. **36** (2015), no. 3, 455–481.

[6] A. O. L. Atkin and J. Lehner, *Hecke operators on* $\Gamma_0(M)$, Math. Ann. **185** (1970), 134–160.

[7] H. Atobe, *Applications of Arthur's multiplicity formula to Siegel modular forms*, https://arxiv.org/abs/1810.09089v1

[8] A. Balog and K. Ono, *The Chebotarev density theorem in short intervals and some questions of Serre*, J. Number Theory **91** (2001), 356–371.

[9] G. Bantle and F. Grupp, *On a problem of Erdös and Szemerédi*, J. Number Theory **22** (1986), 280–288.

[10] T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 1, 29–98.

[11] S. Böcherer, *Bemerkungen über die Dirichletreihen von Koecher und Maass*, Mathematica Gottingensis **68** (1986), 36 pp.

[12] S. Breulmann, *On Hecke eigenforms in the Maass space*, Math. Z. **232** (1999), no. 3, 527–530.

[13] Y-J. Choie and W. Kohnen, *The first sign change of Fourier coefficients of cusp forms*, Amer. J. Math. **131** (2009), no. 2, 517–543.

[14] H. Cohen and F. Strömberg, Modular forms, A classical approach, Graduate Studies in Mathematics, **179**, *American Mathematical Society*, Providence, RI, 2017.

[15] S. Das and S. Ganguly, *Gaps between non-zero Fourier coefficients of cusp forms*, Proc. Amer. Math. Soc. **142** (2014), 3747–3755.

[16] S. Das and J. Sengupta, *An Omega-result for Saito-Kurokawa lifts*, Proc. Amer. Math. Soc. **142** (2014), no. 3, 761–764.

[17] S. Das and W. Kohnen, *On sign changes of eigenvalues of Siegel cusp forms of genus 2 in prime powers*, Acta Arith. **183** (2018), no. 2, 167–172.

[18] S. Das, W. Kohnen and J. Sengupta, *On a convolution series attached to a Siegel Hecke cusp form of degree 2*, Ramanujan J. **33** (2014), no. 3, 367–378.

[19] P. Deligne, *La conjecture de Weil : I*, Publ. Math. I.H.E.S. **43** (1974), 273–307.

[20] M. Eichler and D. Zagier, The theory of Jacobi forms. Birkhäuser 1985.

[21] N. Elkies, *Distribution of supersingular primes*, In Journées Arithmétiques (1989), Astérisque, (1992), 127–132.

[22] P. Erdös, *On the difference of consecutive terms of sequences defined by divisibility properties*, Acta Arith. **12** (1966/1967), 175–182.

[23] M. Filaseta and O. Trifonov, *On gaps between squarefree numbers II*, J. London Math. Soc. **45** (1992), 215–221.

[24] A. Ghosh and P. Sarnak, *Real zeros of holomorphic Hecke cusp forms*, J. Eur. Math. Soc. (JEMS) **14** (2012), no. 2, 465–487.

[25] A. Granville, *ABC allows us to count squarefrees*, Internat. Math. Res. Notices **19** (1998), 991–1009.

[26] S. Gun, W. Kohnen and B. Paul, *Arithmetic behaviour of Hecke eigenvalues of Siegel cusp forms of degree two*, Submitted.

[27] S. Gun, W. Kohnen and P. Rath, *Simultaneous sign change of Fourier coefficients of two cusp forms*, Arch. Math. (Basel) **105** (2015), no. 5, 413–424.

[28] S. Gun, B. Kumar and B. Paul, *The first simultaneous sign change and non-vanishing of Hecke eigenvalues of newforms*, To appear in J. Number Theory.

[29] S. Gun and M. R. Murty, *Generalization of an identity of Ramanujan*, J. Ramanujan Math. Soc. **31** (2016), no. 2, 125–135.

[30] S. Gun and B. Paul, *Sign changes of Fourier coefficients of newforms and multiplicity one theorem*, Submitted.

[31] S. Gun, B. Paul and J. Sengupta, *On Hecke eigenvalues of Siegel modular forms in the Maass space*, Forum. Math. **30** (2018), 775–783.

[32] M. Harris, *Galois representations, automorphic forms, and the Sato-Tate conjecture*, Indian J. Pure Appl. Math. **45** (2014), no. 5, 707–746.

[33] E. Hecke, *Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen*, Zweite Mitteilung, Math. Z. **6** (1920), 11–51;

reprinted in Mathematische Werke, Vandenhoeck & Ruprecht, Göttingen, 1959, 249–289.

[34] A. E. Ingham, The distribution of prime numbers, *Cambridge University Press*, Cambridge, 1990.

[35] H. Iwaniec and E. Kowalski, Analytic Number Theory, *American Mathematical Society*, Providence, RI, 2004.

[36] H. Iwaniec, W. Kohnen and J. Sengupta, *The first negative Hecke eigenvalue*, Int. J. Number Theory **3** (2007), 355–363.

[37] W. Kohnen, *On Hecke eigenvalues of newforms*, Math. Ann. **329** (2004), no. 4, 623–628.

[38] W. Kohnen and J. Sengupta, *On the first sign change of Hecke eigenvalues of newforms*, Math. Z. **254** (2006), 173–184.

[39] W. Kohnen and J. Sengupta, *Signs of Fourier coefficients of two cusp forms of different weights*, Proc. Amer. Math. Soc. **137** (2009), no. 11, 3563–3567.

[40] W. Kohnen and Y. Martin, *Sign changes of Fourier coefficients of cusp forms supported on prime power indices*, Int. J. Number Theory **10** (2014), no. 8, 1921–1927.

[41] E. Kowalski, Y. K. Lau, K. Soundararajan and J. Wu, *On modular signs*, Math. Proc. Cambridge Philos. Soc. **149** (2010), no. 3, 389–411.

[42] E. Kowalski, P. Michel and J. Vanderkam, *Rankin-Selberg L-functions in the level aspect*, Duke Mathematical Journal **114** (2002), 123–191.

[43] E. Kowalski, O. Robert and J. Wu, *Small gaps in coefficients of L-functions and $\mathfrak{B}$-free numbers in short intervals*, Rev. Mat. Iberoamericana, **23** (2007), no. 1, 281–326.

[44] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley, New York, 1974.

[45] M. Kumari and M. R. Murty, *Simultaneous non-vanishing and sign changes of Fourier coefficients of modular forms*, Int. J. Number Theory **14** (2018), no. 8, 2291–2301.

[46] N. Kurokawa, *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two*, Inv. Math. **49** (1978), 149–165.

[47] S. Lang, Linear Algebra, Second edition, *Addison-Wesley Publishing Co.*, 1971.

[48] Y-K. Lau, J. Liu and J. Wu, *Sign changes of the coefficients of automorphic L-functions*, Number Theory: Arithmetic in Shangri-La, edited by S. Kanemitsu, H. Li, and J. Liu, 141–181. Hackensack, NJ: World Scientific Publishing Co. Pvt. Ltd., 2013.

[49] D. H. Lehmer, *The Vanishing of Ramanujan's Function* $\tau(n)$, Duke Math. J. **14** (1947), 429–433.

[50] W-C W. Li, *L-series of Rankin type and their functional equations*, Math. Ann. **244** (1979), 135–166.

[51] W. Luo and D. Ramakrishnan, *Determination of modular forms by twists of critical L-values*, Invent. Math. **130** (1997), no. 2, 371–398.

[52] W. Luo, *Special L-values of Rankin-Selberg convolutions*, Math. Ann. **314** (1999), no. 3, 591–600.

[53] G. Lü, *Sums of absolute values of cusp form coefficients and their application*, J. Number Theory, **139** (2014), 29–43.

[54] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades*, Inv. Math. **52** (1979), 95–104.

[55] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades, II*, Inv. Math. **53** (1979), 249 – 253.

[56] H. Maass, *Über eine Spezialschar von Modulformen zweiten Grades, III*, Inv. Math. **53** (1979), 255 – 265.

[57] K. Matomäki, *On signs of Fourier coefficients of cusp forms*, Math. Proc. Cambridge Philos. Soc. **152** (2012), no. 2, 207–222.

[58] K. Matomäki, *Real zeros of holomorphic Hecke cusp forms and sieving short intervals*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 1, 123–146.

[59] K. Matomäki and M. Radziwiłł, *Sign changes of Hecke eigenvalues*, Geom. Funct. Anal. **25** (2015), 1937–1955.

[60] J. Meher and M. R. Murty, *Sign changes of Fourier coefficients of half-integral weight cusp forms*, Int. J. Number Theory, **10** (2014), No. 4, 905–914.

[61] T. Miyake, Modular Forms, Springer Monographs in Mathematics, *Springer-Verlag*, Berlin, 2006.

[62] J. L. Mordell, *On Mr. Ramanujan's empirical expansions of modular functions*, Proc. Cambridge Philos. Soc. **19** (1917), 117–124.

[63] K. Murty, *Lacunarity of modular forms*, J. Indian Math. Soc. (N.S.) **52** (1987), 127–146 (1988).

[64] M. R. Murty, *Oscillations of Fourier coefficients of modular forms*, Math. Annalen **262** (1983), no 4, 431–446.

[65] M. R. Murty, *Congruences between modular forms*, Analytic Number Theory, Kyoto, (1996), 309–320, London Math., Soc., Lecture Notes, Ser., 247, Cambridge University Press, Cambridge, (1997).

[66] M. R. Murty, Problems in analytic number theory, Graduate Texts in Mathematics **206**, *Springer-Verlag*, New York, 2001.

[67] M. R. Murty and V. K. Murty, Non-vanishing of L-functions and applications, *Modern Birkhäuser Classics*, Birkhäuser/Springer Basel AG, Basel, 1997.

[68] M. R. Murty and V. K. Murty, *Odd values of Fourier coefficients of certain modular forms*, Int. J. Number Theory **3** (2007), no. 3, 455–470.

[69] M. R. Murty and S. Pujahari, *Distinguishing Hecke eigenforms*, Proc. Amer. Math. Soc. **145** (2017), no 5, 1899–1904.

[70] A. P. Ogg, *On a convolution of L-series*, Invent. Math. **7** (1969), 297–312.

[71] T. Oda, *On the poles of Andrianov L-functions*, Math. Ann. **256** (1981), no. 3, 323–340.

[72] V. M. Patankar and C. S. Rajan, *Distinguishing Galois representations by their normalized traces*, J. Number Theory **178** (2017), 118–125.

[73] A. Pitale, *Classical interpretation of the Ramanujan conjecture for Siegel cusp forms of genus n*, Manuscripta Math. **130** (2009), no. 2, 225–231.

[74] A. Pitale, A. Saha and R. Schmidt, *Transfer of Siegel cusp forms of degree 2*, Mem. Amer. Math. Soc. **232** (2014), no. 1090, vi+107.

[75] A. Pitale and R. Schmidt, *Ramanujan-type results for Siegel cusp forms of degree 2*, J. Ramanujan Math. Soc. **24** (2009), no. 1, 87–111.

[76] H. Rademacher, *On the Phragmén-Lindelöf theorem and some applications*, Math. Z. **72** (1959), 192–204.

[77] C. S. Rajan, *Distribution of values of Hecke characters of infinite order*, Acta Arith. **85** (1998), no. 3, 279–291.

[78] D. Ramakrishnan, Appendix: *a refinement of the strong multiplicity one theorem for $GL(2)$*, Invent. Math. **116** (1994), 645–649.

[79] S. Ramanujan, *On certain arithmetical functions*, [Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184]. Collected papers of Srinivasa Ramanujan, 136–162, AMS Chelsea Publ., Providence, RI, 2000.

[80] R. A. Rankin, *Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions II.* Proc. Cambridge Philos. Soc. **35** (1939), 357–372.

[81] R. A. Rankin, *An Ω-result for the coefficients of cusp forms*, Math. Ann. **203** (1973), 239–250.

[82] K. A. Ribet, *Galois representations attached to eigenforms with Nebentypus*, in Modular functions of one variable, V (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 17–51. Lecture Notes in Math. **601**, Springer, Berlin, 1977.

[83] K. A. Ribet, *Twists of modular forms and endomorphisms of abelian varieties*, Math. Ann. **253** (1980), no. 1, 43–62.

[84] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.

[85] E. Royer, J. Sengupta and J. Wu, *Non-vanishing and sign changes of Hecke eigenvalues for Siegel cusp forms of genus two*. With an appendix by E. Kowalski and A. Saha, Ramanujan J. **39** (2016), no. 1, 179–199.

[86] A. Saha, *A relation between multiplicity one and Böcherer's conjecture*, Ramanujan J. **33** (2014), no. 2, 263–268.

[87] P. Sargos, and J. Wu, *Multiple exponential sums with monomials and their applications in number theory*, Acta Math. Hungar. **87** (2000), 333–354.

[88] R. Schmidt, *Archimedean aspects of Siegel modular forms of degree 2*, Rocky Mountain J. Math. **47** (2017), 2381–2422.

[89] R. Schmidt, *On Siegel modular forms of degree* 2 *with square-free level*, Surikaisekikenkyusho Kokyuroku (Research Institute for Mathematical Sciences, Kyoto University) **1338** (2003), 155–169.

[90] J. Sengupta, *Distinguishing Hecke eigenvalues of primitive cusp forms*, Acta Arith. **114** (2004), no. 1, 23–34.

[91] J-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Publ. Math. I.H.E.S. **54** (1981), 323–401.

[92] J-P. Serre, *Sur la lacunarité des puissances de* η, Glasgow Math. J. **27** (1985), 203–221.

[93] J-P. Serre, *Sur les représentations modulaires de degré* 2 *de Gal*$(\overline{\mathbb{Q}}/\mathbb{Q})$, Duke Math. J. **54** (1987), no. 1, 179–230.

[94] G. Shimura, Introduction to the arithmetic theory of automorphic functions, Kanô Memorial Lectures, No. 1, Publications of the Mathematical Society of Japan, Princeton University Press, Princeton, N.J., 1971.

[95] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. (2) **36** (1935), no. 3, 527–606.

[96] C. L. Siegel, *Einführung in die Theorie der Modulfunktionen* n*-ten Grades*, Math. Ann. **116** (1939), 617–657.

[97] E. Szemerédi, *On the difference of consecutive terms of sequences defined by divisibility properties. II*, Acta Arith. **23** (1973), 359–361.

[98] G. Tenenbaum, Introduction to analytic and probabilistic number theory, Cambridge Studies in Advanced Mathematics **46**, *Cambridge University Press*, Cambridge, 1995.

[99] R. Weissauer, Endoscopy for GSp(4) and the cohomology of Siegel modular threefolds, Lecture Notes in Mathematics, **vol. 1968**, *Springer-Verlag*, Berlin, 2009.

[100] J. Wu, *Nombres $\mathfrak{B}$-libres dans les petits intervalles*, Acta Arith. **65** (1993), 97–116.

[101] J. Wu and Y. Ye, *Hypothesis H and the prime number theorem for automorphic representations*, Funct. Approx. Comment. Math. **XXXVII.2** (2007), 461–471.

[102] J. Wu and W. Zhai, *Distribution of Hecke eigenvalues of newforms in short intervals*, Q. J. Math., **64**(2) (2013), 619–644.

[103] D. B. Zagier, *Sur la conjecture de Saito-Kurokawa (d'après H. Maass)*, Sém. Delange-Pisot-Poitou 1979-1980. Prog. Math. **12** (1981), 371–394.

[104] W. G. Zhai, *Number of $\mathfrak{B}$-free numbers in short intervals*, Chinese Sci. Bull. **45** (2000), 208–212.