PERIOD-INDEX PROBLEM FOR HYPERELLIPTIC CURVES

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with an appendix by S. RAMANAN

ABSTRACT. Let C be a smooth projective curve of genus 2 over a number field k with a rational point. We prove that the index and exponent coincide for elements in the 2-torsion of $Ш(Br(C))$. In the appendix, an isomorphism of the moduli space of rank 2 stable vector bundles with odd determinant on a smooth projective hyperelliptic curve C of genus g with a rational point over any field of characteristic not two with the Grassmannian of $(g-1)$-dimensional linear subspaces in the base locus of a certain pencil of quadrics is established, making a result of ([10]) rational. We establish a twisted version of this isomorphism and we derive as a consequence a weak Hasse principle for the smooth intersection X of two quadrics in $\mathbb{P}^5$ over a number field: if X contains a line locally, then X has a k-rational point.

Let k be a field and $\mathrm{Br}(k)$ the Brauer group of k. There are two numerical invariants attached to a Brauer class $\alpha \in \mathrm{Br}(k)$; $index(\alpha) = \sqrt{[D:k]}$ if α is represented by a central division algebra D over k and $period(\alpha) =$ order of α in $\mathrm{Br}(k)$. There has been extensive study of uniform bounds for index of algebras in terms of their periods over fields which are of arithmetic or geometric interest ([9], [17], [32], [15]). Let p be a prime and K a field of characteristic not equal to p. The Brauer p-dimension of K denoted by $\mathrm{Br}_p\mathrm{dim}(K)$ is the least integer d such that for every finite extension L/K and every $\alpha \in {}_p\mathrm{Br}(L)$, $\mathrm{index}(\alpha)$ divides p^d. Here ${}_p\mathrm{Br}(L)$ denotes the p-torsion subgroup of $\mathrm{Br}(L)$.

Bounding Brauer dimension has deep consequences in the study of homogeneous spaces under connected linear algebraic groups. A theorem of de-Jong/Lieblich that period = index for function fields of surfaces over algebraically closed fields is critical to the solution of Conjecture II of Serre for exceptional groups of type D_4, E_6, E_7 due to Gille ([13, IV.2]). Bounding the Brauer 2-dimension of function fields of all curves over totally imaginary number fields would lead to finiteness of the u-invariant of such fields ([20]). Finiteness of the u-invariant of $k(t)$, k totally imaginary number field, is an open question.

Let k be a totally imaginary field with the ring of integers $\mathscr{O}$. Let C/k be a smooth projective geometrically integral curve over k with $\mathscr{C}/\mathscr{O}$ a regular proper model. In ([20]), finiteness of $\mathrm{Br}_p\mathrm{dim}(k(C))$ for all C/k is reduced to bounding indices of p-torsion elements in $\mathrm{Br}(\mathscr{C})$, i.e. 'unramified elements' in $\mathrm{Br}(k(C))$, for all C/k. By a theorem of Grothendieck, for a smooth projective curve C over k, the image of $\mathrm{Br}(\mathscr{C})$ is zero in $\mathrm{Br}(k_\nu(C))$ for every finite place ν of k. Let $Ш\,\mathrm{Br}(k(C)) = Ker(H^2(k(C), \mathbb{G}_m) \to \prod_{\nu\in\Omega_K} H^2(k_\nu(C), \mathbb{G}_m))$. Note that $Ш\,\mathrm{Br}(k(C)) \subseteq \mathrm{Br}(\mathscr{C}) \subseteq \mathrm{Br}(C)$ ([29]) and $Ш\,\mathrm{Br}(k(C)) = \mathrm{Br}(\mathscr{C})$ if k is a totally imaginary number field. We state the following conjecture concerning the period/index bounds for elements in $Ш\,\mathrm{Br}(k(C))$:

Conjecture. *For elements in $Ш\,\mathrm{Br}(k(C))$, the index and period coincide.*